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Density-matrix approach to dynamics of multilevel atoms in laser fields

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Abstract

The theoretical foundations of atom dynamics in laser fields are reviewed in relation with applications to laser spectroscopy, control of atomic motion, atom traps and frequency standards. We present an ab initio approach to the description of internal and translational dynamics of multilevel atoms in laser fields based on the equations for the atomic density matrix. Semiclassical density matrix equations are reviewed and applied to the description of properties of atomic populations and coherences for a classically moving atom. Quantum-kinetic equations for the atomic density matrix are reviewed for the multilevel interaction schemes. The procedure of reduction of the quantum-kinetic equations to the Fokker–Planck quasiclassical kinetic equation for the atomic distribution function is described. Quasiclassical kinetic equations are applied to the multilevel atomic schemes to describe the translational atomic dynamics. Basic types of the dipole radiation forces on atoms are considered for realistic cases of multilevel dipole interaction schemes. The applications of the theory of atomic dynamics in laser fields to the laser cooling, magneto-optical and optical dipole traps, and optical lattices are discussed. © 2002 Published by Elsevier Science B.V.

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1. Introduction

The purpose of this review is to describe the density matrix approach to atomic motion in laser fields, present theoretical fundamentals of translational dynamics of atoms in laser fields, and outline the applications of theoretical approaches to laser control of atomic motion, including laser cooling of atoms and atom traps. For the past decades both internal and translational dynamics of atoms in laser fields have been investigated for many specific dipole interaction schemes and under different conditions. Extensive theoretical and experimental studies of atomic dynamics resulted in the development of the effective techniques to control both the internal and translational atomic states. Among such techniques one can mention optical pumping, velocity-selective excitation of atoms, coherent population trapping and methods of cooling and trapping atoms, deflection, reflection and splitting atomic beams, and guiding atoms in laser fields. It can nowadays be said that the development of the above methods resulted in the creation of foundations of atom manipulation with laser light and atom optics (Letokhov and Chebotayev, 1977; Demtröder, 1996; Minogin and Letokhov, 1987; Kazantsev et al., 1990; Arimondo et al., 1992; Berman, 1997; Grimm et al., 1999; Metcalf and van der Straten, 1999; Balykin et al., 2000).

From a general physical point of view both the internal and translational dynamics of an atom in a laser field can be attributed to one of the two basic types according to the relation between the contributions of the induced and spontaneous transitions. At short interaction time τ_{int} compared with the spontaneous decay times, $\tau_{\mathrm{int}} \ll \tau_{\mathrm{sp}}$, spontaneous transitions cannot play a noticeable role in atomic dynamics. In this relatively simple pure quantum-mechanical case the atomic dynamics is mostly a coherent one, well defined by the time evolution of the initial atom state and initial shape of the atom wavepacket. This case is of basic importance for the coherent atom control by pulsed laser fields and for atom optics. Quite a different and most complicated case occurs when the interaction time is of the order of or exceeds the characteristic relaxation times defined by the spontaneous decays, $\tau_{\mathrm{int}} \gtrsim \tau_{\mathrm{sp}}$. In this most frequently investigated case atomic transitions induced by a laser field are interrupted by a stochastic process of spontaneous photon emission. As a result, spontaneous decays lead to a relaxation of the internal atom states to the quasi-stationary states while the quantum-statistical fluctuations in atomic momentum cause the atomic wave packet to perform a stochastic motion and drift in the momentum space. This latter case of quantum-statistical atom dynamics is of importance for applications related with spectroscopic studies of atoms and control of atomic motion by continuous laser fields.

In this paper we concentrate on the quantum-statistical atom dynamics paying basic attention to the excitation processes and dynamics for multilevel interaction schemes. While internal and translational dynamics of a two-level atom is relatively simple, the dynamics of multilevel atoms exhibits many new features specific of multilevel interaction schemes.

Among studies of the problems of translational dynamics of multilevel atoms in the laser fields the most important is the quantum-statistical approach based on the quantum-kinetic equations for the atomic density matrix. This general approach can be applied to any specific dipole interaction scheme between an atom and the laser field. Depending on the level of simplification this approach can give a simple description of the internal atomic states in the framework of the semiclassical approach, or describe the time evolution of the internal and translational atomic state in terms of quasiclassical approach, and finally give the most complete description of atomic dynamics in terms of a fully quantum-kinetic approach.

This paper aims at the review of basic physical principles of atomic dynamics with applications to basic schemes of laser cooling and trapping of multilevel atoms. We discuss atomic dynamics for practically important laser field configurations and the multilevel dipole interaction schemes relevant to the experiments in the field. The review considers three basic levels of the theoretical description of atomic dynamics in laser fields. First, a relatively simple semiclassical approach is used for describing the internal atomic dynamics and dipole radiation forces on atoms. This approach treats atoms as classically moving systems possessing quantized internal states (Sections 2 and 3). The most general quantum-statistical description in terms of the quantum-kinetic equations for the atomic density matrix is given in Section 4. The quasiclassical level of description is discussed first in general in Section 5 and is applied later in Sections 6–8 for description of laser cooling of atoms and atomic motion in the atom traps.

2. Semiclassical atomic density matrix

In many conventional situations the dipole interaction of an atom with a laser field can be treated assuming that the atomic center of mass moves classically. This assumption is always justified if the change in atomic momentum caused by the photon recoil associated with the absorption or emission of the laser field photons and emission of the vacuum field photons may be considered unimportant under the conditions of a specific problem. In such a case the only effect of the dipole interaction of the atom with a laser field is an excitation of a classically moving atom at the internal transitions. In this case the internal atomic dynamics is described by the semiclassical atomic density matrix which parametrically depends on the classical coordinates of the atomic center of mass. In this section we introduce the concept of the semiclassical atomic density matrix and describe the specific forms of the density matrix equations for the multilevel interaction schemes. The material presented below represents a particular case of application of a general quantum-statistical concept of the density matrix to a case of the electric dipole interaction of an atom with the electromagnetic field (Neumann, 1955; Fano, 1957; Haar, 1961; Landau and Lifshitz, 1977; Blum, 1981).

2.1. Dynamic equations for a motionless atom

We first recall the basic notions on the atomic density matrix for a case when a motionless atom interacts with a laser field at short times compared with the spontaneous decay times, $\tau_{\rm int} \ll \tau_{\rm sp}$. In this case the dynamics of the internal atomic states is completely described by the Schrödinger equation with the atomic wave function $\Psi = \Psi(\xi, t)$ which depends on the set of coordinates $\xi = \xi_1, \xi_2, \ldots, \xi_n$

used for the description of the internal atomic motion, i.e. the motion of the electrons and nuclei in the atom. The atomic wave function is assumed to be normalized "per a single atom".

Under the dipole interaction of a motionless atom with a classical laser field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ the Hamiltonian H includes a proper atomic Hamiltonian $H_\mathrm{a} = H_\mathrm{a}(\xi)$ which depends on the internal atomic coordinates ξ, and the dipole interaction operator $V = -\mathbf{d}\cdot\mathbf{E}$,

$$H = H_\mathrm{a} + V = H_\mathrm{a} - \mathbf{d}\cdot\mathbf{E}\ , \tag{1}$$

where $\mathbf{d} = \mathbf{d}(\xi)$ is the operator of the electric dipole moment of the atom or simply the electric dipole operator. Proper atomic Hamiltonian is assumed to possess the atomic energy eigenvalues E_n and eigenfunctions $\Psi_n(\xi, t)$,

$$\Psi_n(\xi, t) = \psi_n(\xi)\mathrm{e}^{-\mathrm{i}E_n t/\hbar},$$

$$H_\mathrm{a}\psi_n(\xi) = E_n\psi_n(\xi)\ , \tag{2}$$

describing the stationary states of the atom in the absence of the dipole perturbation V. The atomic eigenfunctions $\psi_n(\xi)$ are assumed to satisfy the orthonormalization conditions,

$$\int \psi_m^*(\xi)\psi_n(\xi)\,\mathrm{d}^3\xi = \delta_{mn}\ . \tag{3}$$

Note that we use a simplified notation for the differential taken over the internal coordinates, $\mathrm{d}^3\xi = \mathrm{d}^3\xi_1 \ldots \mathrm{d}^3\xi_n$.

The decomposition of the atomic wave function over the time-dependent eigenfunctions of a proper atomic Hamiltonian

$$\Psi(\xi, t) = \sum a_k \Psi_k(\xi, t)\ , \tag{4}$$

gives the decomposition of the atomic density matrix considered as a function of atomic coordinates ξ and ξ' as

$$\rho(\xi, \xi', t) = \Psi(\xi, t)\Psi^*(\xi', t) = \sum_{m,n} \rho_{mn}\psi_m(\xi)\psi_n^*(\xi')\mathrm{e}^{-\mathrm{i}(E_m - E_n)t/\hbar}\ , \tag{5}$$

where the atomic density matrix elements defined with respect to the time-dependent eigenfunctions are

$$\rho_{mn} = a_m a_n^*\ . \tag{6}$$

The atomic density matrix function satisfies the equation of motion

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\rho(\xi, \xi', t) = (H(\xi) - H^*(\xi'))\rho(\xi, \xi', t)\ , \tag{7}$$

where it is assumed that the Hamiltonian $H(\xi)$ acts on coordinates ξ and the Hamiltonian $H(\xi')$ acts on coordinates ξ', and the equations of motion for the atomic density matrix elements are

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\rho_{kl} = \sum_m V_{km}(t)\rho_{ml} - \sum_n \rho_{kn}V_{nl}(t)\ . \tag{8}$$

In the above interaction representation

$$V_{kl}(t) = \int \Psi_k^*(\xi, t)V(\xi)\Psi_l(\xi, t)\,\mathrm{d}^3\xi = V_{kl}\mathrm{e}^{\mathrm{i}\omega_{kl}t} \tag{9}$$

are the matrix elements of the dipole interaction operator defined with respect to the time-dependent eigenfunctions and $\omega_{kl} = (E_k - E_l)/\hbar$. The quantities

$$V_{kl} = \int \psi_k^*(\xi) V(\xi) \psi_l(\xi)\, \mathrm{d}^3\xi = -\mathbf{d}_{kl} \cdot \mathbf{E} \tag{10}$$

are the matrix elements of the interaction operator taken with respect to the time-independent eigenfunctions and

$$\mathbf{d}_{kl} = \int \psi_k^*(\xi) \mathbf{d}(\xi) \psi_l(\xi)\, \mathrm{d}^3\xi \tag{11}$$

are the matrix elements of the atomic dipole moment.

Decomposition of the atomic wave function over the time-independent eigenfunctions $\psi_k(\xi)$ of a proper atomic Hamiltonian,

$$\Psi(\xi, t) = \sum \tilde{a}_k \psi_k(\xi) \tag{12}$$

gives the decomposition of the atomic density matrix function as

$$\rho(\xi, \xi', t) = \sum_{m,n} \tilde{\rho}_{mn} \psi_m(\xi) \psi_n^*(\xi') \ . \tag{13}$$

In the same interaction representation, the density matrix elements defined with respect to the time-independent eigenfunctions, $\tilde{\rho}_{mn} = \tilde{a}_m \tilde{a}_n^*$, are related to the density matrix elements defined with respect to the time-dependent eigenfunctions, as

$$\tilde{\rho}_{mn} = \rho_{mn} \mathrm{e}^{-\mathrm{i}\omega_{mn} t} \ .$$

The equations of motion for the density matrix elements $\tilde{\rho}_{mn}$,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \tilde{\rho}_{kl} = (E_k - E_l)\tilde{\rho}_{kl} + \sum_m V_{km} \tilde{\rho}_{ml} - \sum_n \tilde{\rho}_{kn} V_{nl} \ , \tag{14}$$

differ from Eqs. (8) by the additional energy terms.

2.2. Dynamic equations for a moving atom

2.2.1. Galileo transformation

The structure of the semiclassical density matrix equations for a classically moving atom can always be found by a transformation from the atom rest frame to the laboratory frame. As before we neglect here the spontaneous relaxation and consider only the dynamic terms in the semiclassical density matrix.

Assume that the atomic center of mass has coordinates $\mathbf{r}, t$ in the laboratory frame and coordinates $\mathbf{r}', t'$ in the atom rest frame. For simplicity, the two reference frames may be considered to coincide at the initial instant of time $t = t' = 0$. At any arbitrary instant of time the coordinates of the atom in the two reference frames are connected by the Galileo transformation,

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t', \quad t = t' \ ,$$

where $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity of the atom in the laboratory frame.

In Eqs. (8) and (14) the time derivative is considered to be taken in the atom frame, i.e. as $\partial/\partial t'$. Changing the variables from $\mathbf{r}'$ and t' to $\mathbf{r}$ and t according to the inverse Galileo transformation,

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t, \quad t' = t \ ,$$

one can express the time derivative in the atom frame through the derivatives in the laboratory frame,

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial \mathbf{r}}{\partial t'}\frac{\partial}{\partial \mathbf{r}} = \frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} \ .$$

The last equation represents a general result. The density matrix equations for a moving atom include not the partial but the total (or convective) time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} \ . \tag{15}$$

The total time derivative describes the evolution of the atomic density matrix both in time and space.

The transformation to the laboratory frame thus shows that for a classically moving atom the density matrix equations differ from Eqs. (8) and (14) by a meaning of the time derivative only. For a moving atom the equations for the elements of the density matrix ρ are

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho_{kl} = -\sum_m (\mathbf{d}_{km}\cdot\mathbf{E})\mathrm{e}^{\mathrm{i}\omega_{km}t}\rho_{ml} + \sum_n \rho_{kn}(\mathbf{d}_{nl}\cdot\mathbf{E})\mathrm{e}^{\mathrm{i}\omega_{nl}t} \ , \tag{16}$$

and for the elements of the density matrix $\tilde{\rho}$ are

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_{kl} = (E_k - E_l)\tilde{\rho}_{kl} - \sum_m (\mathbf{d}_{km}\cdot\mathbf{E})\tilde{\rho}_{ml} + \sum_n \tilde{\rho}_{kn}(\mathbf{d}_{nl}\cdot\mathbf{E}) \ . \tag{17}$$

In the above equations the density matrix elements are functions of a space coordinate and time and parametrically depend on the atomic velocity, $\rho_{kl} = \rho_{kl}(\mathbf{r},\mathbf{v},t)$, $\tilde{\rho}_{kl} = \tilde{\rho}_{kl}(\mathbf{r},\mathbf{v},t)$. The laser field is considered as taken at the position of the atom, $\mathbf{E} = \mathbf{E}(\mathbf{r},t)$.

2.2.2. Rotating wave approximation

When a multilevel atom interacts with the laser field composed of monochromatic waves, say, plane travelling waves,

$$\mathbf{E} = \sum_a (\mathbf{E}^a \mathrm{e}^{\mathrm{i}(\mathbf{k}_a\mathbf{r}-\omega_a t)} + \mathbf{E}^{a*}\mathrm{e}^{-\mathrm{i}(\mathbf{k}_a\mathbf{r}-\omega_a t)}) \ , \tag{18}$$

near resonant to the atomic transitions with frequencies $\omega_{mn} = (E_m - E_n)/\hbar > 0$, the "fast" terms oscillating at frequencies $\omega_a + \omega_{mn} \approx 2\omega_a$ (twice the optical frequencies ω_a) can be neglected compared with the "slow" terms oscillating at different frequencies $|\omega_a - \omega_{mn}| \ll \omega_a$ (see, e.g., Allen and Eberly, 1975). In this rotating wave approximation (RWA) the dynamic terms in the density matrix equations include only slowly varying terms. Eq. (16) defining the density matrix ρ can be written in RWA as

$$\begin{aligned}\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho_{kl} = &-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^a)\rho_{ml}\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{km})t} + \sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^a)\rho_{kn}\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{nl})t} \\ &- \sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^{a*})\rho_{ml}\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{mk})t} + \sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^{a*})\rho_{kn}\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{ln})t} \ .\end{aligned} \tag{19}$$

In the above form of writing it is assumed that the equations include the terms with positive atomic transition frequencies only. The first sum is assumed to include the terms with frequencies $\omega_{km} = (E_k - E_m)/\hbar > 0$, the second sum with frequencies $\omega_{nl} > 0$, the third sum with frequencies $\omega_{mk} > 0$, and the fourth sum with frequencies $\omega_{ln} > 0$. The quantities $\delta_{mn} = \omega_a - \omega_{mn}$ entering Eqs. (19) have accordingly the meaning of the detunings of the laser field frequencies ω_a with respect to the atomic transition frequencies $\omega_{mn} > 0$.

2.2.3. Dipole interaction matrix elements

For an arbitrarily polarized laser field (18) the dipole interaction terms can be evaluated by decomposing the dipole moment $\mathbf{d}$ and the field amplitudes $\mathbf{E}^a, \mathbf{E}^{a*}$ over the spherical unit vectors

$$\mathbf{e}_0 = \mathbf{e}_z, \quad \mathbf{e}_\pm = \mp\frac{1}{\sqrt{2}}(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y) . \tag{20}$$

With the definition for the spherical vector components d_0, $d_\pm$ and E_0^a, $E_\pm^a$ identical to that for the spherical unit vectors (20) the decompositions are

$$\mathbf{d} = d_0\mathbf{e}_0 - d_-\mathbf{e}_+ - d_+\mathbf{e}_- ,$$

$$\mathbf{E}^a = E_0^a\mathbf{e}_0 - E_-^a\mathbf{e}_+ - E_+^a\mathbf{e}_- .$$

This gives for the scalar product

$$\mathbf{d}\cdot\mathbf{E}^a = d_0E_0^a - d_-E_+^a - d_+E_-^a ,$$

and for the matrix elements of the dipole interaction terms

$$\mathbf{d}_{kl}\cdot\mathbf{E}^a = \langle k|\mathbf{d}|l\rangle\cdot\mathbf{E}^a = \langle k|d_0|l\rangle E_0^a - \langle k|d_-|l\rangle E_+^a - \langle k|d_+|l\rangle E_-^a . \tag{21}$$

The matrix elements $(d_q)_{kl} = \langle k|d_q|l\rangle$, $q = 0, \pm 1$, of the dipole moment spherical components can usually be expressed through the reduced dipole matrix element $\langle k\|d\|l\rangle$. In typical dipole interaction schemes the atomic states are described by the angular momentum states $|k\rangle = |\alpha jm\rangle$. In such cases the dependence of the dipole matrix elements on the magnetic quantum numbers can be found with the use of the Wigner–Eckart theorem (see, e.g., Edmonds, 1974; Zare, 1988)

$$\langle\alpha' j' m'|d_q|\alpha jm\rangle = (-1)^{j'-m'}\begin{pmatrix} j' & 1 & j \\ -m' & q & m \end{pmatrix}\langle\alpha' j'\|d\|\alpha j\rangle , \tag{22}$$

where $\langle\alpha' j'\|d\|\alpha j\rangle$ is the reduced dipole matrix element.

In applications related to the control of atomic motion by the laser fields of basic interest are the dipole interaction schemes which include the hyperfine structure states $|\alpha FM\rangle$, where $\alpha \equiv nLSJI$ denotes the quantum numbers of the fine structure states. For such schemes the dipole matrix elements can be evaluated as

$$\langle\alpha' F' M'|d_q|\alpha FM\rangle = (-1)^{F'-M'}\begin{pmatrix} F' & 1 & F \\ -M' & q & M \end{pmatrix}\langle\alpha' F'\|d\|\alpha F\rangle , \tag{23}$$

where the reduced matrix element for the hyperfine structure transition is expressed through the reduced matrix element $\langle\alpha'\|d\|\alpha\rangle$ for the fine structure transition,

$$\langle\alpha' F'\|d\|\alpha F\rangle = (-1)^{J'+I+F+1}\sqrt{(2F+1)(2F'+1)}\begin{Bmatrix} J' & F' & I \\ F & J & 1 \end{Bmatrix}\langle\alpha'\|d\|\alpha\rangle\ . \tag{24}$$

The values of the $3-j$ and $6-j$ symbols entering Eqs. (22)–(24) can be found for specific atomic schemes in Edmonds (1974), Sobelman (1979) and Zare (1988).

2.3. Spontaneous relaxation terms and complete semiclassical equations

The spontaneous relaxation terms in the equations for the semiclassical density matrix follow from the consideration of the dipole interaction of the atom with a vacuum electromagnetic field treated as a quantized photon field. The basic lines of derivation are explicitly shown below for the simplest case of a two-level atom. The relaxation terms for the multilevel atomic schemes are given below for a sufficiently general and practically most important case of the hyperfine structure states.

2.3.1. Quantized vacuum field

We first recall the field quantization procedure needed for the consideration below (Heitler, 1944; Berestetskii et al., 1971; Louisell, 1973; Loudon, 1983). The quantization procedure treats a free vacuum field as a stationary system of quantum-mechanical harmonic oscillators described by the Hamiltonian $H_{\rm v}$,

$$H_{\rm v} = \sum \hbar\omega_\lambda(a_\lambda a_\lambda^+ + 1/2)\ , \tag{25}$$

where the annihilation a_λ and creation a_λ^+ operators satisfy the commutation relations

$$[a_\lambda, a_{\lambda'}] = [a_\lambda^+, a_{\lambda'}^+] = 0, \quad [a_\lambda, a_{\lambda'}^+] = \delta_{\lambda\lambda'}$$

and index $\lambda = (\mathbf{k}, i)$ specifies a particular field oscillator with wave vector $\mathbf{k}$ and polarization $i = 1, 2$, and $\omega_\lambda = kc$.

Each stationary state of a set of harmonic oscillators is described by the wave function $\Phi = \Phi_{n_1 n_2 n_3 \ldots}$ which is a product of the stationary wave functions φ_{n_λ} of the field oscillators,

$$\Phi_{n_1 n_2 n_3 \ldots} = \prod_{n_\lambda} \varphi_{n_\lambda}\ , \tag{26}$$

where n_λ is the number of photons of a type $\lambda = (\mathbf{k}, i)$ in a vacuum field. The partial wave functions $\varphi_{n_\lambda} = |n_\lambda\rangle$ satisfy the usual orthonormalization conditions

$$\langle\varphi_{n_\lambda}|\varphi_{n_\mu}\rangle = \langle n_\lambda|n_\mu\rangle = \delta_{\lambda\mu}\ . \tag{27}$$

The action of the annihilation and creation operators on the field oscillator wave function $\varphi_{n_\lambda} = |n_\lambda\rangle$ is defined by the expressions which follow from the equations for the matrix elements of the harmonic oscillator

$$\begin{aligned} a_\lambda|n_\lambda\rangle &= \sqrt{n_\lambda}|n_\lambda - 1\rangle\ , \\ a_\lambda^+|n_\lambda\rangle &= \sqrt{n_\lambda + 1}|n_\lambda + 1\rangle\ , \\ a_\lambda^+ a_\lambda|n_\lambda\rangle &= n_\lambda|n_\lambda\rangle\ , \end{aligned} \tag{28}$$

where the last expression jointly with orthonormalization condition (27) define the number of photons n_λ in the mode $\lambda = (\mathbf{k}, i)$ of a vacuum field,

$$\langle \Phi_{n_1 n_2 n_3 \ldots} | a_\lambda^+ a_\lambda | \Phi_{n_1 n_2 n_3 \ldots} \rangle = \langle n_\lambda | a_\lambda^+ a_\lambda | n_\lambda \rangle = \langle n_\lambda | n_\lambda | n_\lambda \rangle = n_\lambda \ . \tag{29}$$

The energy of a quantized vacuum field according to the above equations is

$$E_\mathrm{v} = \langle \Phi_{n_1 n_2 n_3 \ldots} | H_\mathrm{v} | \Phi_{n_1 n_2 n_3 \ldots} \rangle = \sum \hbar \omega_\lambda (n_\lambda + \tfrac{1}{2}) \ . \tag{30}$$

The consideration of the vacuum field variables as operators makes both the vector potential and the electric and magnetic fields to become operators as well. In terms of the annihilation and creation operators the operator of the vector potential $\mathbf{A}$ and the operators of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are

$$\begin{aligned} \mathbf{A} &= \sum (\mathbb{A}_\lambda a_\lambda + \mathbb{A}_\lambda^* a_\lambda^\dagger) \ , \\ \mathbf{E} &= \sum (\mathbb{E}_\lambda a_\lambda + \mathbb{E}_\lambda^* a_\lambda^\dagger) \ , \\ \mathbf{B} &= \sum (\mathbb{B}_\lambda a_\lambda + \mathbb{B}_\lambda^* a_\lambda^\dagger) \ , \end{aligned} \tag{31}$$

where $\mathbb{A}_\lambda$, $\mathbb{E}_\lambda$ and $\mathbb{B}_\lambda$ are the vector potential, electric and magnetic field of a "single photon",

$$\begin{aligned} \mathbb{A}_\lambda &= c \sqrt{\frac{2\pi\hbar}{\mathrm{V}\omega_\lambda}} \, \mathbf{e}_\lambda \mathrm{e}^{\mathrm{i}kr} \ , \\ \mathbb{E}_\lambda &= \mathrm{i} \sqrt{\frac{2\pi\hbar\omega_\lambda}{\mathrm{V}}} \mathbf{e}_\lambda \mathrm{e}^{\mathrm{i}kr} \ , \\ \mathbb{B}_\lambda &= \mathrm{i}c \sqrt{\frac{2\pi\hbar}{\mathrm{V}\omega_\lambda}} [\mathbf{k} \times \mathbf{e}_\lambda] \mathrm{e}^{\mathrm{i}kr} \ , \end{aligned} \tag{32}$$

and $\mathbf{e}_\lambda$ ($\lambda = (\mathbf{k}, i)$) is a unit vector defining the polarization of a plane travelling wave with the wave vector $\mathbf{k}$, and V is the quantization volume. The above relations fully define a vacuum field as a quantum-mechanical system.

2.3.2. Two-level atom

To maintain formal similarity with the quantum-mechanical description of the vacuum field and a two-level atom one can introduce the atomic lowering b and rising b^+ operators defined by the equations

$$b|g\rangle = 0, \quad b|e\rangle = |g\rangle, \quad b^+|g\rangle = |e\rangle, \quad b^+|e\rangle = 0 \ . \tag{33}$$

With these definitions the Hamiltonian for a motionless two-level atom can be written as

$$H_\mathrm{a} = \hbar\omega_0 b^+ b \ . \tag{34}$$

In accordance with Eqs. (33) the eigenvalues of the atomic Hamiltonian H_a are defined here assuming the ground state to have zero energy,

$$E_g = \langle g | H_\mathrm{a} | g \rangle = 0, \quad E_e = \langle e | H_\mathrm{a} | e \rangle = \hbar\omega_0 \ . \tag{35}$$

The matrix elements of the dipole moment operator $\hat{\mathbf{d}}$ for a two-level atom can be chosen to be real,

$$\langle g|\hat{\mathbf{d}}|g\rangle = \langle e|\hat{\mathbf{d}}|e\rangle = 0, \quad \langle e|\hat{\mathbf{d}}|g\rangle = \langle g|\hat{\mathbf{d}}|e\rangle = \mathbf{d} \ . \tag{36}$$

The last equations being compared with Eqs. (33) show that the atomic dipole moment operator is represented in terms of the atomic operators b and b^+ as

$$\hat{\mathbf{d}} = \mathbf{d}(b + b^+) \ . \tag{37}$$

With the above definitions the dipole interaction operator $\tilde{V}$ responsible for the interaction of a two-level atom with the vacuum field can be written in RWA as

$$\tilde{V} = -\mathbf{d} \cdot \sum (\mathbb{E}_\lambda b^+ a_\lambda + \mathbb{E}^*_\lambda b a^\dagger_\lambda) \ , \tag{38}$$

where the summation includes all the modes of the vacuum field. The matrix elements of the RWA interaction operator other than zero are only for the one-photon absorption or emission processes,

$$\begin{aligned} \tilde{V}_{eg;n_\lambda,n_\lambda+1} &= \langle e, n_\lambda|\tilde{V}|g, n_\lambda + 1\rangle = -\mathbf{d} \cdot \mathbb{E}_\lambda \sqrt{n_\lambda + 1} \ , \\ \tilde{V}_{ge;n_\lambda+1,n_\lambda} &= \langle g, n_\lambda + 1|\tilde{V}|e, n_\lambda\rangle = -\mathbf{d} \cdot \mathbb{E}^*_\lambda \sqrt{n_\lambda + 1} \ . \end{aligned} \tag{39}$$

Considering the Schrödinger equation with the Hamiltonian that includes the vacuum field Hamiltonian, proper atomic Hamiltonian and the dipole interaction operator,

$$H = \sum \hbar\omega_\lambda (a_\lambda a^+_\lambda + \tfrac{1}{2}) + \hbar\omega_0 b^+ b - \mathbf{d} \cdot \sum (\mathbb{E}_\lambda b^+ a_\lambda + \mathbb{E}^*_\lambda b a^\dagger_\lambda) \ , \tag{40}$$

one can derive a set of equations for the probability amplitudes c_{k,n_λ} which describe the joint states of a two-level atom and the vacuum field. To do so, the total wave function for an "atom+vacuum field" system is to be decomposed over the eigenfunctions of a two-level atom and the vacuum field,

$$\Psi = \sum_{k\lambda} c_{k,n_\lambda} \psi_k \mathrm{e}^{-\mathrm{i}E_k t/\hbar} \prod_\lambda \varphi_{n_\lambda} \mathrm{e}^{-\mathrm{i}E_\lambda t/\hbar} \ . \tag{41}$$

It is to be noted here that the probability amplitudes c_{k,n_λ} depend on all the occupation numbers n_λ. Using for shortness a single index n_λ in Eq. (41) we assume here that the quantity n_λ describes the manifold of the quantum occupation numbers, $n_\lambda \equiv \{n_\lambda\} = \{n_1, n_2, \ldots, n_\lambda \ldots\}$, where n_1 is the number of photons in the vacuum mode 1, n_2 is the number of photons in the vacuum mode 2,..., and n_λ is number of photons in the vacuum mode λ. It is to be also kept in mind that the atomic eigenfunctions ψ_k satisfy the eigenvalue equations

$$H_\mathrm{a}\psi_n = E_n\psi_n$$

with eigenvalues (35), and the vacuum field eigenfunctions satisfy the equations

$$H_\mathrm{v}\varphi_{n_\lambda} = E_\lambda \varphi_{n_\lambda}, \quad E_\lambda = \hbar\omega_\lambda(n_\lambda + \tfrac{1}{2}) \ .$$

Substituting the decomposition (41) into the Schrödinger equation with Hamiltonian (40) one can get a set of equations for the probability amplitudes c_{e,n_λ} and c_{g,n_λ} :

$$\begin{aligned} \mathrm{i}\dot{c}_{e,n_\lambda} &= -\frac{1}{\hbar} \sum_\lambda \mathbf{d} \cdot \mathbb{E}_\lambda \sqrt{n_\lambda + 1}\, \mathrm{e}^{-\mathrm{i}\Delta_\lambda t} c_{g,n_\lambda+1} \ , \\ \mathrm{i}\dot{c}_{g,n_\lambda+1} &= -\frac{1}{\hbar} \sum_\lambda \mathbf{d} \cdot \mathbb{E}^*_\lambda \sqrt{n_\lambda + 1}\, \mathrm{e}^{\mathrm{i}\Delta_\lambda t} c_{e,n_\lambda} \ , \end{aligned} \tag{42}$$

where $\Delta_\lambda = \omega_\lambda - \omega_0$ is the detuning of the vacuum mode frequency ω_λ with respect to the atomic transition frequency ω_0.

An infinite set of equations (42) describes the quantized states of a closed system "atom+vacuum field". Taking the products of the probability amplitudes according to Eq. (6) one can introduce the density matrix for the "atom+vacuum field" system defined by the density matrix elements

$$\rho_{kl;n_\lambda n_\mu} = c_{k,n_\lambda} c^*_{l,n_\mu} \ . \tag{43}$$

The density matrix with elements $\rho_{kl;n_\lambda n_\mu}$ describes both the state of the atom (indices k, l) and the states of the vacuum field (indices n_λ and n_μ). Taking a trace over the vacuum field occupation numbers one can introduce the semiclassical atomic density matrix ρ_{kl}:

$$\begin{aligned} \rho_{ee} &= \sum_\lambda c_{e,n_\lambda} c^*_{e,n_\lambda} \ , \\ \rho_{eg} &= \sum_\lambda c_{e,n_\lambda} c^*_{g,n_\lambda+1} \ , \\ \rho_{gg} &= \sum_\lambda c_{g,n_\lambda+1} c^*_{g,n_\lambda+1} \ , \end{aligned} \tag{44}$$

which satisfies the usual Hermiticity condition, $\rho_{ge} = \rho^*_{eg}$.

Note now that the density matrix elements (43) should in general be averaged over an infinite number of vacuum modes (of which each includes an infinite number of photon states). The averaging procedure can be considerably simplified if one follows the Weisskopf–Wigner theory of the spontaneous emission which states that an explicit form of the relaxation terms does not depend on the specific state of a vacuum field (Weisskopf and Wigner, 1930; Agarwal, 1974). According to this important idea one can choose the simplest state of the vacuum field corresponding to zero occupation numbers for all the vacuum modes. A two-level atom can accordingly be considered as initially occupying the excited state $|e\rangle$.

One may thus assume that at the initial instant of time $t = t_i$ the "atom+vacuum field" system occupies a single state $|e, 0\rangle$ with the probability amplitude $c_{e,0}(t_i) = 1$, and all the other initial probability amplitudes are equal to zero. In particular, the initial probability amplitudes for the states $|g, 1_\lambda\rangle$ into which the atom could emit a photon are also equal to zero, $c_{g,1_\lambda}(t_i) = 0$. Under this simplest choice of the initial conditions the equations for the probability amplitudes describing the interaction of the atom with a vacuum field are:

$$\begin{aligned} \dot{c}_{e,0} &= \frac{\mathrm{i}}{\hbar} \sum_\lambda \mathbf{d} \cdot \mathbb{E}_\lambda \mathrm{e}^{-\mathrm{i}\Delta_\lambda t} c_{g,1_\lambda} \ , \\ \dot{c}_{g,1_\lambda} &= \frac{\mathrm{i}}{\hbar} \mathbf{d} \cdot \mathbb{E}^*_\lambda \mathrm{e}^{\mathrm{i}\Delta_\lambda t} c_{e,0} \ . \end{aligned} \tag{45}$$

Here the first equation describes the spontaneous decay of the atom into all the modes of the vacuum field. The second equation describes an excitation of a partial vacuum mode due to the spontaneous decay. The probability amplitudes entering Eqs. (45) define the semiclassical atomic density matrix

elements as

$$\begin{aligned}\rho_{ee} &= c_{e,0}c^*_{e,0}\ ,\\ \rho_{eg} &= \sum_\lambda c_{e,0}c^*_{g,1_\lambda} = c_{e,0}\sum_\lambda c^*_{g,1_\lambda}\ ,\\ \rho_{gg} &= \sum_\lambda c_{g,1_\lambda}c^*_{g,1_\lambda}\ .\end{aligned} \tag{46}$$

The above relations can now be used to find the relaxation terms in the equations for the semiclassical atomic density matrix. Taking the formal solution of the second equation in (45),

$$c_{g,1_\lambda} = \frac{\mathrm{i}}{\hbar}\,\mathbf{d}\cdot\mathbb{E}^*_\lambda \int_{t_i}^{t} \mathrm{e}^{\mathrm{i}\Delta_\lambda t'} c_{e,0}(t')\,\mathrm{d}t'\ ,$$

and substituting it into the first equation (45) one gets

$$\dot{c}_{e,0} = -\frac{1}{\hbar^2}\sum_\lambda |\mathbf{d}\cdot\mathbb{E}_\lambda|^2 \int_{t_i}^{t} \mathrm{e}^{\mathrm{i}\Delta_\lambda(t'-t)} c_{e,0}(t')\,\mathrm{d}t'\ . \tag{47}$$

The sum taken in the last equation over all vacuum modes can be replaced by an integral over the wave vectors $\mathbf{k}$. Taking into account the fact that a single vacuum mode occupies in the wave vector space a "volume"

$$(\Delta k)^3 = (2\pi/L)^3 = (2\pi)^3/\mathrm{V}\ ,$$

where $\mathrm{V} = L^3$, one can rewrite the above replacement as

$$\sum_\lambda \to \frac{\mathrm{V}}{(2\pi)^3}\int \mathrm{d}^3k\ .$$

Next, one can make a transition to the integral over the frequencies $\omega = kc$ and the wave vector directions ($\mathrm{d}o = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$),

$$\frac{\mathrm{V}}{(2\pi)^3}\int \mathrm{d}^3k \to \frac{\mathrm{V}}{(2\pi)^3}\int k^2\,\mathrm{d}k\,\mathrm{d}o = \frac{\mathrm{V}}{(2\pi c)^3}\int \omega^2\,\mathrm{d}\omega\,\mathrm{d}o\ .$$

Note next that according to the energy conservation law the frequencies of the emitted photons are to be very close to the atomic transition frequency, $\omega \simeq \omega_0$, while the probability amplitudes $c_{e,0}(t')$ are the slowly varying functions of the frequency ω. Under the last assumptions the integral over the frequencies is reduced to a well-known formula

$$\int \mathrm{e}^{\mathrm{i}(\omega-\omega_0)(t'-t)}\,\mathrm{d}\omega = 2\pi\delta(t-t')\ .$$

The integration over time reduces the integral in Eq. (47) to the final integral over the solid angle. Assuming that the direction of the quantization axis Oz is defined by the direction of the vector $\mathbb{E}_\lambda$ one can write the scalar product in Eq. (47) as $\mathbf{d}\cdot\mathbb{E}_\lambda = |\mathbf{d}|\,|\mathbb{E}_\lambda|\cos\theta$ and get for the final integral $\int\cos^2\theta\,\mathrm{d}o = 4\pi/3$. This reduces Eq. (47) to

$$\dot{c}_{e,0} = -\gamma c_{e,0}\ , \tag{48}$$

where γ is half of the spontaneous decay rate,

$$2\gamma = W_{\rm sp} = \frac{4|\mathbf{d}|^2\omega_0^3}{3\hbar c^3} \ . \tag{49}$$

Taking the time derivative of the first equation in (46) and making use of Eq. (48) one finally gets the equation describing the spontaneous decay of the upper-state density matrix element,

$$\frac{\partial}{\partial t}\,\rho_{ee} = -2\gamma\rho_{ee} \ , \tag{50}$$

Taking the time derivative of the second equation in (46) and noting that the second term of the derivative decays to zero according to the first equation in (45) one gets the equation describing the spontaneous decay of the off-diagonal density matrix element,

$$\frac{\partial}{\partial t}\,\rho_{eg} = -\gamma\rho_{eg} \ . \tag{51}$$

For the rate of change of the ground-state density matrix element ρ_{gg} one can get by the above method the equation

$$\frac{\partial}{\partial t}\,\rho_{gg} = 2\gamma\rho_{ee} \ . \tag{52}$$

The validity of the last equation can also be seen from Eq. (50) and the normalization condition for the semiclassical atomic density matrix elements, $\rho_{gg} + \rho_{ee} = 1$.

The above spontaneous relaxation terms being added to the dynamic terms completely define the semiclassical density matrix equations for both motionless and moving two-level atoms since the relaxation terms are invariant under the Galileo transformations. Complete semiclassical equations for the atomic density matrix describing the interaction of a classically moving two-level atom with a classical light field $\mathbf{E}$ can be finally written by collecting together the dynamic and stochastic terms (50)–(52). For the basic case of interaction of a two-level atom with a monochromatic plane travelling laser wave,

$$\mathbf{E} = \tfrac{1}{2}\,\mathbf{E}_0(\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)} + \mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\omega t)}) = \mathbf{E}_0\cos(\mathbf{kr} - \omega t) \ , \tag{53}$$

the RWA equations are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_{ee} &= \mathrm{i}\Omega(\rho_{ge}\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)} - \rho_{eg}\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\delta t)}) - 2\gamma\rho_{ee} \ , \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{eg} &= \mathrm{i}\Omega(\rho_{gg} - \rho_{ee})\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)} - \gamma\rho_{eg} \ , \\
\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_{gg} &= \mathrm{i}\Omega(\rho_{eg}\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\delta t)} - \rho_{ge}\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)}) + 2\gamma\rho_{ee} \ ,
\end{aligned} \tag{54}$$

where the Rabi frequency is defined as $\Omega = \mathbf{d}\cdot\mathbf{E}/2\hbar$ and $\delta = \omega - \omega_0$ is the detuning of the laser wave frequency with respect to the atomic transition frequency.

2.3.3. Multilevel atom

The spontaneous relaxation terms $(\Gamma\rho)_{kl} = \langle k|\Gamma\rho(\mathbf{r},\mathbf{p},t)|l\rangle$ coming from the dipole interaction of a multilevel atom with a vacuum field can be found by the same Weisskopf–Wigner procedure

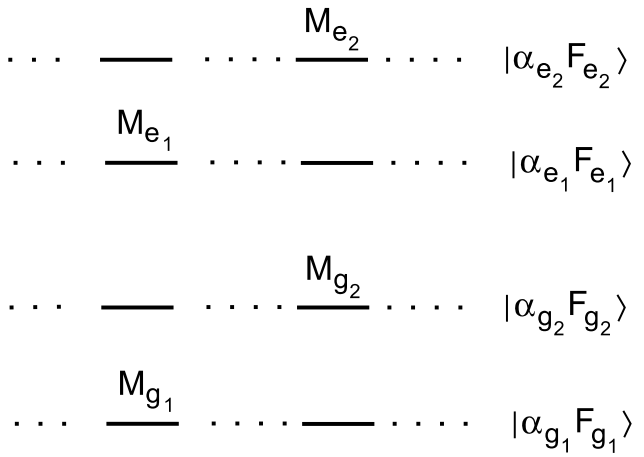

Fig. 1. Multilevel atomic energy scheme consisting of the ground-state magnetic sublevels $|\alpha_g F_g M_g\rangle$ and the excited-state magnetic sublevels $|\alpha_e F_e M_e\rangle$.

considered above for a two-level atom. By adding these terms to the dynamic terms one can obtain a complete equation for the semiclassical atomic density matrix,

$$i\hbar \frac{d}{dt} \rho_{kl} = \text{Dynamic Terms} + i\hbar \langle k|\Gamma\rho|l\rangle \ . \tag{55}$$

Below we specify the spontaneous relaxation terms for the dipole interaction schemes which include the hyperfine structure states $|\alpha FM\rangle$, where $\alpha \equiv nLSJI$ denotes all the quantum numbers of the fine structure atomic states. For such schemes the ground-state magnetic sublevels are denoted as $|\alpha_g F_g M_g\rangle$ and the excited-state magnetic sublevels as $|\alpha_e F_e M_e\rangle$ (Fig. 1). With these notations the spontaneous relaxation terms entering the semiclassical equations (55) for the atomic density matrix elements $\rho_{kl} = \rho_{\alpha_a F_a M_a, \alpha_b F_b M_b} = \langle \alpha_a F_a M_a|\rho|\alpha_b F_b M_b\rangle$ can be shown to be (Happer, 1972; Ducloy, 1973; Cohen-Tannoudji, 1977; Omont, 1977; Rautian and Shalagin, 1991):

$$\begin{aligned}
&\langle \alpha_{e_1} F_{e_1} M_{e_1}|\Gamma\rho|\alpha_{e_2} F_{e_2} M_{e_2}\rangle = -(\gamma_{\alpha_{e_1} F_{e_1}} + \gamma_{\alpha_{e_2} F_{e_2}})\langle \alpha_{e_1} F_{e_1} M_{e_1}|\rho|\alpha_{e_2} F_{e_2} M_{e_2}\rangle \ , \\
&\langle \alpha_e F_e M_e|\Gamma\rho|\alpha_g F_g M_g\rangle = -\gamma_{\alpha_e F_e}\langle \alpha_e F_e M_e|\rho|\alpha_g F_g M_g\rangle \ , \\
&\langle \alpha_{g_1} F_{g_1} M_{g_1}|\Gamma\rho|\alpha_{g_2} F_{g_2} M_{g_2}\rangle \\
&\quad = \sum_{\alpha_{e_1}, \alpha_{e_2}, F_{e_1}, F_{e_2}, M_{e_1}, M_{e_2}} (F_{g_1} F_{g_2} M_{g_1} M_{g_2}|A|F_{e_1} F_{e_2} M_{e_1} M_{e_2})\langle \alpha_{e_2} F_{e_2} M_{e_2}|\rho|\alpha_{e_1} F_{e_1} M_{e_1}\rangle \ , \\
&\langle \alpha_g F_g M'_g|\Gamma\rho|\alpha_g F_g M_g\rangle = \sum_{\alpha_e F_e M_e M'_e} (F_g M_g M'_g|A|F_e M_e M'_e)\langle \alpha_e F_e M'_e|\rho|\alpha_e F_e M_e\rangle \ ,
\end{aligned} \tag{56}$$

where the coefficients $(F_{g_1} F_{g_2} M_{g_1} M_{g_2}|A|F_{e_1} F_{e_2} M_{e_1} M_{e_2})$ symbolically including the Einstein coefficient A define the joint relaxation of two magnetic substates. These coefficients can be expressed through

the Clebsch–Gordan coefficients $(F_g M_g M'_g|A|F_e M_e M'_e)$ according to the equations:

$$
\begin{aligned}
&(F_{g_1} F_{g_2} M_{g_1} M_{g_2}|A|F_{e_1} F_{e_2} M_{e_1} M_{e_2}) \\
&\quad = (\gamma_{\alpha_{e_1} F_{e_1}, \alpha_{g_1} F_{g_1}} + \gamma_{\alpha_{e_2} F_{e_2}, \alpha_{g_2} F_{g_2}}) \sum_{q=0,\pm 1} (F_{g_1} M_{g_1} 1q|F_{e_1} M_{e_1})(F_{g_2} M_{g_2} 1q|F_{e_2} M_{e_2}) \,, \\
&(F_g M_g M'_g|A|F_e M_e M'_e) = 2\gamma_{\alpha_e F_e, \alpha_g F_g} \sum_{q=0,\pm 1} (F_g M'_g 1q|F_e M'_e)(F_g M_g 1q|F_e M_e) \,.
\end{aligned} \tag{57}
$$

The right-hand sides of the above equations can also be expressed through the $3-j$ symbols by making use of standard relations between the Clebsch–Gordan coefficients and the $3-j$ symbols,

$$
(F_g M_g 1q|F_e M_e) = (-1)^{F_g - 1 + M_e} \sqrt{2F_e + 1} \begin{pmatrix} F_g & 1 & F_e \\ M_g & q & -M_e \end{pmatrix} .
$$

Note that in the above formulas the magnetic quantum numbers satisfy the selection rules following from the properties of the Clebsch–Gordan coefficients,

$$
\begin{aligned}
&M_{e_1} - M_{e_2} = M_{g_1} - M_{g_2} \,, \\
&M_e - M'_e = M_g - M'_g \,.
\end{aligned} \tag{58}
$$

The partial spontaneous decay rate from the excited state $|\alpha_e F_e\rangle$ to the ground state $|\alpha_g F_g\rangle$ is

$$
2\gamma_{\alpha_e F_e, \alpha_g F_g} = W_{\mathrm{sp}}(F_e \to F_g) = \frac{4}{3} \frac{|\langle \alpha_e F_e \| d \| \alpha_g F_g \rangle|^2 \omega_{eg}^3}{(2F_e + 1)\hbar c^3} \,, \tag{59}
$$

where $\langle \alpha_e F_e \| d \| \alpha_g F_g \rangle$ is a reduced dipole matrix element for a hyperfine structure transition. The total spontaneous decay rate from the hyperfine structure state $|\alpha_e F_e\rangle$ to all the hyperfine states $|\alpha_g F_g\rangle$ belonging to the ground state is

$$
2\gamma_{\alpha_e F_e} = W_{\mathrm{sp}}(F_e) = 2 \sum_{\alpha_g F_g} \gamma_{\alpha_e F_e, \alpha_g F_g} \,. \tag{60}
$$

The reduced dipole matrix element $\langle \alpha_e F_e \| d \| \alpha_g F_g \rangle$ can also be expressed through the reduced dipole matrix element $\langle \alpha_e \| d \| \alpha_g \rangle$ for the fine structure transition defined by the quantum numbers $\alpha \equiv nLSJI$ (see, e.g. Edmonds, 1974; Sobelman, 1979). This gives an expression for the spontaneous decay rate between two hyperfine structure states as

$$
2\gamma_{\alpha_e F_e, \alpha_g F_g} = W_{\mathrm{sp}}(F_e \to F_g) = (2J_e + 1)(2F_g + 1) \begin{Bmatrix} J_e & F_e & I \\ F_g & J_g & 1 \end{Bmatrix}^2 W_{\mathrm{sp}}(\alpha_g \to \alpha_e) \,,
$$

$$
W_{\mathrm{sp}}(\alpha_g \to \alpha_e) = \frac{4}{3} \frac{|\langle \alpha_e \| d \| \alpha_g \rangle|^2 \omega_{eg}^3}{(2J_e + 1)\hbar c^3} \,,
$$

where $W_{\mathrm{sp}}(\alpha_g \to \alpha_e)$ is the spontaneous decay rate at the fine structure transition $|\alpha_e\rangle \to |\alpha_g\rangle$ and J_e is the quantum number of the atomic momentum for the excited fine structure state $|\alpha_e\rangle$.

It is worth noting that the spontaneous relaxation terms entering the semiclassical density matrix equations are the same in both forms of the density matrix, i.e. the terms $\langle k|\Gamma\tilde{\rho}|l\rangle$ differ from terms $\langle k|\Gamma\rho|l\rangle$ by the notations for the density matrix only. This is obvious since the interaction of the atom with a vacuum field cannot depend on the form of description of the atomic states.

Finally, for a moving multilevel atom the complete semiclassical equations for the atomic density matrix elements ρ_{kl} in RWA include the dynamic terms entering Eqs. (19) and the spontaneous decay terms (56),

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho_{kl} = -\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^{a})\rho_{ml}\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{km})t} + \sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^{a})\rho_{kn}\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{nl})t} \tag{61}$$

$$-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^{a*})\rho_{ml}\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{mk})t} + \sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^{a*})\rho_{kn}\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{ln})t}$$

$$+\mathrm{i}\hbar\langle k|\Gamma\rho|l\rangle\ . \tag{62}$$

In the above RWA equations the convention used in Eqs. (19) is adopted, all four sums are assumed to include the terms with positive atomic frequencies only, $\omega_{pq}=(E_p-E_q)/\hbar>0$. Similar equations written for the density matrix elements $\tilde{\rho}_{kl}$ include additional energy terms $(E_k-E_l)\tilde{\rho}_{kl}$ as in Eqs. (17).

3. Dipole radiation forces

3.1. Dipole force on a moving atom

3.1.1. General equation

In the framework of the semiclassical description of atomic dynamics when the photon recoil is neglected the laser field cannot directly influence the translational motion of the atom. The field however induces the atomic dipole moment. The interaction of the induced dipole moment with the gradient of the laser field produces according to classical electrodynamics the force on the atom usually called the dipole radiation force.

For a classically moving atom, the induced atomic dipole moment $\langle\mathbf{d}\rangle$ is defined by a usual quantum-statistical mean value,

$$\langle\mathbf{d}\rangle = \mathrm{Tr}(\rho\mathbf{d})\ , \tag{63}$$

where ρ is the semiclassical atomic density matrix. The atom classically moving in a laser field can thus be considered as a classical point-like particle possessing a dipole moment $\langle\mathbf{d}\rangle$. The dipole interaction energy of the atom with a laser field is defined by a mean value

$$U=\langle V\rangle = -\langle\mathbf{d}\rangle\cdot\mathbf{E}\ , \tag{64}$$

where the induced dipole moment and the field are assumed to be taken at the position of the atom center of mass, $\langle\mathbf{d}\rangle=\langle\mathbf{d}\rangle(\mathbf{r},\mathbf{v},t)$, $\mathbf{E}=\mathbf{E}(\mathbf{r},t)$, since a classically moving particle has to be considered to have well-defined coordinate $\mathbf{r}$ and velocity $\mathbf{v}$. According to a classical meaning of Eq. (64) the dipole radiation force on the atom is determined by an expression formally identical to the classical expression for the force on a particle possessing the electric dipole moment (Letokhov and Minogin, 1981; Stenholm, 1986; Minogin and Letokhov, 1987; Cohen-Tannoudji et al., 1992)

$$\mathbf{F}=-\nabla U=\nabla(\langle\mathbf{d}\rangle\cdot\mathbf{E})=\sum\langle d_i\rangle\nabla E_i\ , \tag{65}$$

where index $i=x,y,z$ defines the rectangular vector components of the induced atomic dipole moment and a classical laser field.

Note that when applying formula (65) to any specific dipole interaction scheme one has to take into account two important circumstances. First, the induced dipole moment $\langle \mathbf{d} \rangle$ entering Eq. (65) is to be considered as a constant quantity not to be differentiated on coordinate. Second, since the semiclassical atomic density matrix ρ defining the induced dipole moment $\langle \mathbf{d} \rangle$ is usually considered in a rotating wave approximation (RWA), the right-hand side of Eq. (65) is also to be calculated in RWA. The formula (65) generally defines the dipole radiation force on the atom as a function of atom position $\mathbf{r}$ and velocity $\mathbf{v}$. The position dependence of the force generally originates from the position dependence of the field, $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$, and the position dependence of the atomic density matrix ρ. The velocity dependence of the force comes from the dependence of the atomic density matrix on the velocity.

It is also worth noting that the dipole radiation force, as any other force, is a classical concept. Due to this reason the application of the basic formula (65) to a case of the dipole interaction of a quantized atom with a classical laser field is related to the restrictions under which the atom can be considered as a classically moving particle possessing a well-defined induced dipole moment. For any dipole interaction scheme two basic physical circumstances limit the use of formula (65). First, in order for the mean value of the dipole moment $\langle \mathbf{d} \rangle$ to well characterize the induced atomic dipole moment, the quantum fluctuations of the atomic dipole moment should be small compared with its mean value. Second, in order for the translational motion of the atom to be considered as a classical one, the quantum fluctuations of atomic momentum should be small compared to the mean value of the atomic momentum.

For a simplest model of a two-level atom with allowed dipole transition both the above conditions are satisfied when the time of the dipole interaction between the atom and a laser field, τ_{int}, considerably exceeds the spontaneous decay time τ_{sp}:

$$\tau_{\mathrm{int}} \gg \tau_{\mathrm{sp}} \ . \tag{66}$$

Under condition (66) the spontaneous relaxation leads to a fast relaxation of the internal atomic state to the quasistationary state possessing small fluctuations in the atomic dipole moment. Simultaneously, the quantum fluctuations in atomic momentum become small since at the interaction times (66) the atom scatters a large number of photons. For more complicated multilevel interaction schemes the conditions of a classical motion should be especially investigated for a given scheme. In particular, when the laser field excites the atom from the ground-state sublevels, the ground-state coherence may have a relaxation time that is of the same order as or even much longer than the interaction time between the atom and the field. In such a case the fluctuations of the induced atomic dipole moment can be of the order of the mean value of the atomic dipole moment, and the concept of the force cannot become valid.

3.1.2. Dissipative and reactive (gradient) forces

In typical experimental situations the laser field consists of a number of laser beams. Such a field generally has two characteristic spatial scales. One scale is defined by the wavelength $\lambda = 2\pi/k$ of the laser beams. The second larger scale, $l \gg \lambda$, is defined by the radii of the laser beams. In such a general case the dipole radiation force can be represented as consisting of two parts, the force associated with the gradient of the phases of the laser beams and the force associated with the gradient of the laser beam amplitudes. The decomposition of the dipole radiation force over the above two parts directly follows from general equation (65). In case of field (18) composed

of laser beams, $\mathbf{E}^a = \mathbf{E}^a(\mathbf{r})$, the gradient in Eq. (65) acts on both the phases $\mathbf{k}_a\mathbf{r}$ and amplitudes $\mathbf{E}^a(\mathbf{r})$. This yields gives the dipole radiation force as a sum of two terms, the dissipative and reactive parts,

$$\begin{aligned}
&\mathbf{F} = \mathbf{F}_{\text{diss}} + \mathbf{F}_{\text{react}} \ , \\
&\mathbf{F}_{\text{diss}} = \sum_{a,i} \mathrm{i}(\mathbf{k}_a \langle d_i \rangle E_i^a(\mathbf{r}) \mathrm{e}^{\mathrm{i}(\mathbf{k}_a\mathbf{r} - \omega_a t)} - \text{c.c.}) \ , \\
&\mathbf{F}_{\text{react}} = \sum_{a,i} (\langle d_i \rangle (\nabla E_i^a(\mathbf{r})) \mathrm{e}^{\mathrm{i}(\mathbf{k}_a\mathbf{r} - \omega_a t)} + \text{c.c.}) \ ,
\end{aligned} \tag{67}$$

where index $i = x, y, z$ defines the rectangular vector components. In terms of elementary photon processes the dissipative part of the dipole radiation force, $\mathbf{F}_{\text{diss}}$, comes basically from the absorption of photons by the atom, i.e. from the absorption and subsequent spontaneous emission of photons. This part of the total dipole radiation force is proportional to the photon linear momenta $\hbar\mathbf{k}_a$ and is often called the radiation pressure force. The reactive part of the total force, $\mathbf{F}_{\text{react}}$, comes basically from the induced scattering of photons at the atomic dipole transitions. This part of the dipole radiation force is often called the dipole gradient force or the gradient force.

3.2. Two-level atom in a laser beam

Before discussing the semiclassical dynamics of multilevel atoms in this subsection we present basic results on the semiclassical dynamics of a two-level atom interacting with the near resonant field of a laser beam.

When a two-level atom with ground state $|g\rangle$ and excited state $|e\rangle$ interacts with the field of a monochromatic laser beam defined by a unit polarization vector $\mathbf{e}$, an amplitude $E_0(\mathbf{r})$, and a wave vector $\mathbf{k}$ ($k = \omega/c$),

$$\mathbf{E} = \mathbf{e}E_0(\mathbf{r}) \cos(\mathbf{k}\mathbf{r} - \omega t) \ , \tag{68}$$

the laser beam induces a mean atomic dipole moment

$$\langle \mathbf{d} \rangle = \mathrm{Tr}(\rho\mathbf{d}) = \rho_{ge}\mathbf{d} \exp(\mathrm{i}\omega_0 t) + \rho_{eg}\mathbf{d} \exp(-\mathrm{i}\omega_0 t) \ , \tag{69}$$

where the dipole matrix element is assumed to be real, $\mathbf{d} = \mathbf{d}_{eg} = \mathbf{d}_{ge}$, and $\omega_0 = (E_e - E_g)/\hbar$ is the atomic transition frequency.

3.2.1. Atomic populations and coherences

The semiclassical atomic density matrix elements for a two-level dipole interaction scheme satisfy the RWA equations of motion which differ from Eqs. (54) by a dependence of the Rabi frequency on the atomic coordinate only, $\Omega(\mathbf{r}) = dE_0(\mathbf{r})/2\hbar$, where $d = \mathbf{d} \cdot \mathbf{e}$ is a projection of the dipole matrix element onto the unit polarization vector $\mathbf{e}$. By a substitution for the off-diagonal elements $\rho_{ge} = \sigma_{ge} \exp(-\mathrm{i}(\mathbf{k}\mathbf{r} - \delta t))$ these equations can be reduced to the equations which do not include

"fast" time and position dependence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{ee} &= \mathrm{i}\Omega(\mathbf{r})(\sigma_{ge}-\sigma_{eg}) - 2\gamma\rho_{ee}\ ,\\
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{eg} &= \mathrm{i}\Omega(\mathbf{r})(\rho_{gg}-\rho_{ee}) - [\gamma+\mathrm{i}(\delta-\mathbf{kv})]\sigma_{eg}\ ,\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{gg} &= \mathrm{i}\Omega(\mathbf{r})(\sigma_{eg}-\sigma_{ge}) + 2\gamma\rho_{ee}\ .
\end{aligned} \tag{70}$$

Eqs. (70) jointly with the normalization condition, $\rho_{gg}+\rho_{ee}=1$, fully define the time evolution of a classically moving two-level atom. The steady-state solutions to Eqs. (70) define the atomic populations $n_g=\rho_{gg}$ and $n_e=\rho_{ee}$ and atomic coherences σ_{eg} and $\sigma_{ge}=\sigma_{eg}^*$ as

$$\begin{aligned}
\rho_{ee} &= \frac{\Omega^2(\mathbf{r})}{\gamma^2+2\Omega^2(\mathbf{r})+(\delta-\mathbf{kv})^2}\ ,\\
\rho_{gg} &= \frac{\gamma^2+\Omega^2(\mathbf{r})+(\delta-\mathbf{kv})^2}{\gamma^2+2\Omega^2(\mathbf{r})+(\delta-\mathbf{kv})^2}\ ,\\
\sigma_{eg} &= -\frac{\Omega(\mathbf{r})(\delta-\mathbf{kv}-\mathrm{i}\gamma)}{\gamma^2+2\Omega^2(\mathbf{r})+(\delta-\mathbf{kv})^2}\ .
\end{aligned} \tag{71}$$

3.2.2. Radiation pressure force and gradient force

According to Eq. (69) a mean value of the atomic dipole moment is represented in RWA as

$$\langle\mathbf{d}\rangle = \mathbf{d}(\sigma_{ge}\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\omega t)}+\sigma_{eg}\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)})\ . \tag{72}$$

The dipole radiation force (65) on a two-level atom in the field of a laser beam (68), calculated in RWA, is represented by a sum of two forces (67), the dissipative force which has a meaning of the radiation pressure force $\mathbf{F}_{\mathrm{rp}}$ and the reactive force usually called the gradient force $\mathbf{F}_{\mathrm{gr}}$,

$$\begin{aligned}
&\mathbf{F}=\mathbf{F}_{\mathrm{rp}}+\mathbf{F}_{\mathrm{gr}},\\
&\mathbf{F}_{\mathrm{rp}} = \frac{\mathrm{i}}{2}\langle\mathbf{d}\cdot\mathbf{e}\rangle\mathbf{k}E_0(\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}-\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\omega t)}) = 2\hbar\mathbf{k}\Omega(\mathbf{r})\mathrm{Im}\,\sigma_{eg}\ ,\\
&\mathbf{F}_{\mathrm{gr}} = \tfrac{1}{2}\langle\mathbf{d}\cdot\mathbf{e}\rangle(\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}+\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\omega t)})\nabla E_0(\mathbf{r}) = 2\hbar(\nabla\Omega(\mathbf{r}))\mathrm{Re}\,\sigma_{eg}\ ,
\end{aligned} \tag{73}$$

where according to condition (66) the quantity σ_{eg} is considered as the steady-state solution (71) of the density matrix equations.

In the framework of the semiclassical analysis the radiation pressure force $\mathbf{F}_{\mathrm{rp}}$ originates from the interaction of an induced atomic dipole moment with the laser field varying on space scale $\lambda=2\pi/k$. The gradient force $\mathbf{F}_{\mathrm{gr}}$ comes from the interaction of an induced atomic dipole moment with the field varying on space scale l about the size of the laser beam, $l\simeq|\Omega/\nabla\Omega|$.

The substitution of Eqs. (71) into Eqs. (73) gives the final expressions for the radiation pressure force and gradient force

$$\mathbf{F}_{\mathrm{rp}} = \hbar\mathbf{k}\gamma\,\frac{G(\mathbf{r})}{1+G(\mathbf{r})+(\delta-\mathbf{kv})^2/\gamma^2}\ , \tag{74}$$

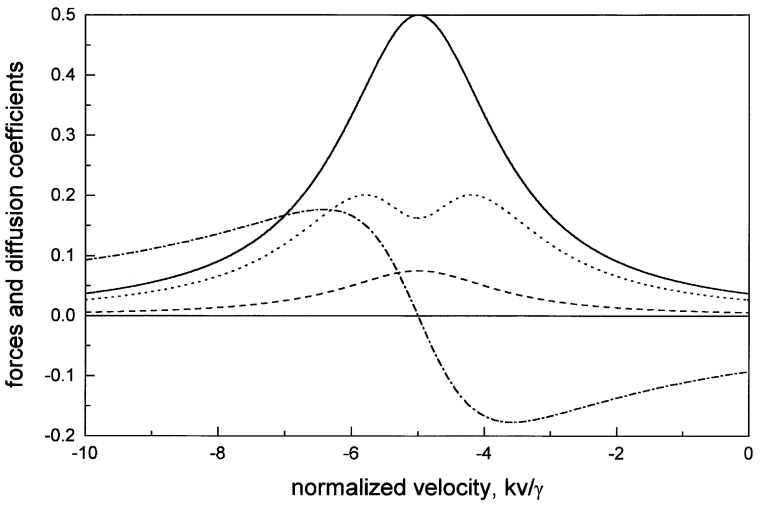

Fig. 2. The velocity dependence of the radiation pressure force $F_{\rm rp}/\hbar k\gamma$ (solid line), gradient force $F_{\rm gr}/\hbar(-\nabla G/G)\gamma$ (dashed-dotted line), transverse $D_{xx}/\hbar^2k^2\gamma$ (dashed line) and longitudinal $D_{zz}/\hbar^2k^2\gamma$ (dotted line) diffusion coefficients for a two-level atom in a laser beam at the detuning $\delta=-5\gamma$ and saturation parameter $G=1$. Note, that quantity $-\nabla G/G$ is positive for a laser beam with a maximum intensity near the axis.

$$\mathbf{F}_{\rm gr}=-\frac{1}{2}\,\hbar(\delta-\mathbf{kv})\frac{\nabla G(\mathbf{r})}{1+G(\mathbf{r})+(\delta-\mathbf{kv})^2/\gamma^2}\ , \tag{75}$$

where

$$G(\mathbf{r})=\frac{2\Omega^2(\mathbf{r})}{\gamma^2}=\frac{1}{2}\left(\frac{dE_0(\mathbf{r})}{\hbar\gamma}\right)^2=\frac{I(\mathbf{r})}{I_{\rm S}} \tag{76}$$

is a dimensionless saturation parameter, $I(\mathbf{r})=(c/8\pi)E_0^2(\mathbf{r})$ is the laser beam intensity at point $\mathbf{r}$, and $I_{\rm S}=\hbar\gamma\omega_0^3/6\pi c^2$ is the saturation intensity. The velocity dependences of the forces $\mathbf{F}_{\rm rp}$ and $\mathbf{F}_{\rm gr}$ reflect the nature of a near resonant atom-laser beam interaction exhibiting resonance properties near velocity $v_{\rm res}=\delta/k$ (Fig. 2). The directions of the radiation pressure force and the gradient force are generally different. The radiation pressure force accelerates or decelerates the atom in the direction of the wave vector $\mathbf{k}$. The direction of the gradient force is defined by the gradient of the laser beam intensity and the value of the difference between the detuning δ and the Doppler shift $\mathbf{kv}$.

It is worth noting that the explicit expressions for the forces allow one to clarify the conditions of validity of Eqs. (74) and (75). The necessity of condition (66) is evident from Eqs. (70) which describe the decay of the atomic functions with characteristic time $\tau_{\rm sp}=\gamma^{-1}$. The second condition discussed in Section 3.1.1 can be shown to give no additional limitations. According to the minimal frequency width of the dipole optical resonance, $\delta\omega\approx\gamma$, the characteristic variation of atomic momentum is defined by a spontaneous decay rate as $\delta p\approx M\gamma/k$. For a classically moving atom this quantity must be considered to exceed the quantum fluctuations of atomic momentum, $\hbar k\ll M\gamma/k$. This gives the second condition for the validity of the classical atomic motion

$$\omega_{\rm r}=\frac{\hbar k^2}{2M}\ll\gamma\ , \tag{77}$$

where $\omega_r = \hbar k^2/2M$ is the frequency defined by the recoil energy R, $\omega_r = R/\hbar$. This second condition is, however, always satisfied for allowed dipole transitions (Minogin and Letokhov, 1987), and hence condition (66) remains the only one needed for the validity of the forces (74) and (75).

The structure of the dipole radiation force on a two-level atom in a laser field composed of a number of the laser beams, including theoretically important standing wave configuration, can be found in (Letokhov et al., 1976, 1977; Stenholm et al., 1978; Minogin and Serimaa, 1979; Gordon and Ashkin, 1980; Minogin and Letokhov, 1987; Minogin and Rozhdestvenskii, 1987).

3.2.3. Optical potential

In applications related to the control of atomic motion, generally both parts of the dipole radiation force influence atomic dynamics. In some cases the action of one part of the total force can be neglected compared with that of the other part. Deceleration, deflection and laser cooling of atoms are based on the action of the radiation pressure force while the gradient force usually plays a negligible role in these processes. On the contrary, the techniques of atom trapping and guiding in laser fields and atom optics are based on the action of the gradient force with small contributions due to the radiation pressure force (Grimm et al., 1999; Balykin et al., 2000). In the latter case the gradient force at a low atomic velocity can be considered as nearly a potential force (Gordon and Ashkin, 1980).

At a low atomic velocity the gradient force (75) can be integrated to give a potential energy often referred to as an optical potential,

$$U_{\rm gr}(\mathbf{r}) = -\int_{-\infty}^{\mathbf{r}} \mathbf{F}_{gr}(\mathbf{v}=0)\cdot d\mathbf{r} = \frac{1}{2}\hbar\delta \ln\left(1+\frac{G(\mathbf{r})}{1+\delta^2/\gamma^2}\right) . \tag{78}$$

For a focused laser beam possessing a maximum intensity at the center of the beam, in the case of red detuning, $\delta < 0$, formula (78) defines a potential well. If the laser beam intensity has a minimum at the center of the beam like the laser mode TEM_{01}^{*} formula (78) defines a potential well at blue detuning, $\delta > 0$. At large detuning, $|\delta| \gg \gamma, \Omega$, the potential (78) of the gradient force is reduced to a simple expression,

$$U_{\rm gr}(\mathbf{r}) = \hbar\,\frac{\Omega^2(\mathbf{r})}{\delta} . \tag{79}$$

The above equation is useful for practical estimations of the depth U_0 of the potential well. In case of a focused laser beam with maximum intensity at the center the potential well existing at $\delta < 0$ has the depth $U_0 = \hbar\Omega^2(0)/|\delta|$. In terms of the quasienergy states (Zel'dovich, 1973) or dressed states (Cohen-Tannoudji et al., 1992) the last equation defines the light shift for the ground state of a two-level atom.

It is worth noting that a simplest model of a dipole interaction of a two-level atom with an inhomogeneous light field (68) can give useful estimations in many specific situations. In particular, the dipole radiation force on a two-level atom in an evanescent laser wave is also defined by the general equations (73). In this case the forces (74) and (75) have specific dependences on atomic position since the spatial dependence of an evanescent wave field differs from that for a laser beam. The field of the evanescent wave decays fast, on a length scale about wavelength, to the vacuum region producing accordingly a considerable gradient force on the atom (Cook and Hill, 1982).

3.3. Multilevel atoms in $\sigma^+ - \sigma^-$ field configuration

Some multilevel atomic schemes frequently used as model schemes are shown in Fig. 3. All these schemes refer to the dipole interaction of the multilevel atoms with the laser light which is near resonant to the atomic transition between two degenerate hyperfine structure states $|\alpha_g F_g\rangle$ and $|\alpha_e F_e\rangle$. For such schemes the dynamic terms (21) in the semiclassical density matrix equations include the dipole matrix elements defined with respect to the nondegenerate states $|\alpha_g F_g M_g\rangle$ and $|\alpha_e F_e M_e\rangle$. These terms can be evaluated using Eqs. (23) and (24). The spontaneous relaxation terms for the above schemes have relatively simple structures following from general equations (56),

$$\begin{aligned}
&\langle F_e M'_e|\Gamma\rho|F_e M_e\rangle = -2\gamma\langle F_e M'_e|\rho|F_e M_e\rangle \ , \\
&\langle F_e M_e|\Gamma\rho|F_g M_g\rangle = -\gamma\langle F_e M_e|\rho|F_g M_g\rangle \ , \\
&\langle F_g M'_g|\Gamma\rho|F_g M_g\rangle = \sum_{M_e M'_e} (F_g M_g M'_g|A|F_e M_e M'_e)\langle F_e M'_e|\rho|F_e M_e\rangle \ ,
\end{aligned} \tag{80}$$

where the incoming terms are defined by the products of the Clebsch–Gordan coefficients,

$$(F_g M_g M'_g|A|F_e M_e M'_e) = 2\gamma \sum_{q=0,\pm1} (F_g M'_g 1q|F_e M'_e)(F_g M_g 1q|F_e M_e) \ ,$$

and the magnetic quantum numbers satisfy the rule,

$$M_e - M'_e = M_g - M'_g \ .$$

In the above equations the spontaneous decay rate from the excited state $|\alpha_e F_e\rangle$ is defined by a standard formula,

$$2\gamma = W_{\rm sp} = \frac{4}{3}\,\frac{|\langle\alpha_e F_e\|d\|\alpha_g F_g\rangle|^2\omega_0^3}{(2F_e+1)\hbar c^3} \ , \tag{81}$$

with $\langle\alpha_e F_e\|d\|\alpha_g F_g\rangle$ being the reduced dipole matrix element for a hyperfine structure transition and ω_0 the atomic transition frequency.

Below we consider some basic examples of the dipole radiation forces on the multilevel atoms which are assumed to interact with a monochromatic laser field composed of two counter-propagating laser waves chosen for definiteness as left circularly polarized waves,

$$\begin{aligned}
&\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \ , \\
&\mathbf{E}_1 = \tfrac{1}{2}E_0(\mathbf{e}_+ e^{{\rm i}(kz-\omega t)} - \mathbf{e}_- e^{-{\rm i}(kz-\omega t)}), \quad \mathbf{E}_2 = \tfrac{1}{2}E_0(-\mathbf{e}_+ e^{{\rm i}(kz+\omega t)} + \mathbf{e}_- e^{-{\rm i}(kz+\omega t)}) \ ,
\end{aligned} \tag{82}$$

where the spherical unit vectors are defined by Eq. (20) and $k = \omega/c$ is the magnitude of the wavevector. With respect to the quantization axis Oz the first wave in Eq. (82) is a σ^+ polarized wave and the second one is a σ^- polarized wave. The field (82) is often referred to as a $\sigma^+ - \sigma^-$ laser field configuration.

All the examples presented below are given for practically important atomic schemes with integer total angular momentum. Everywhere below the detuning is defined as for a two-level atom, $\delta = \omega - \omega_0$, while the Rabi frequency Ω is defined separately for every atomic model.

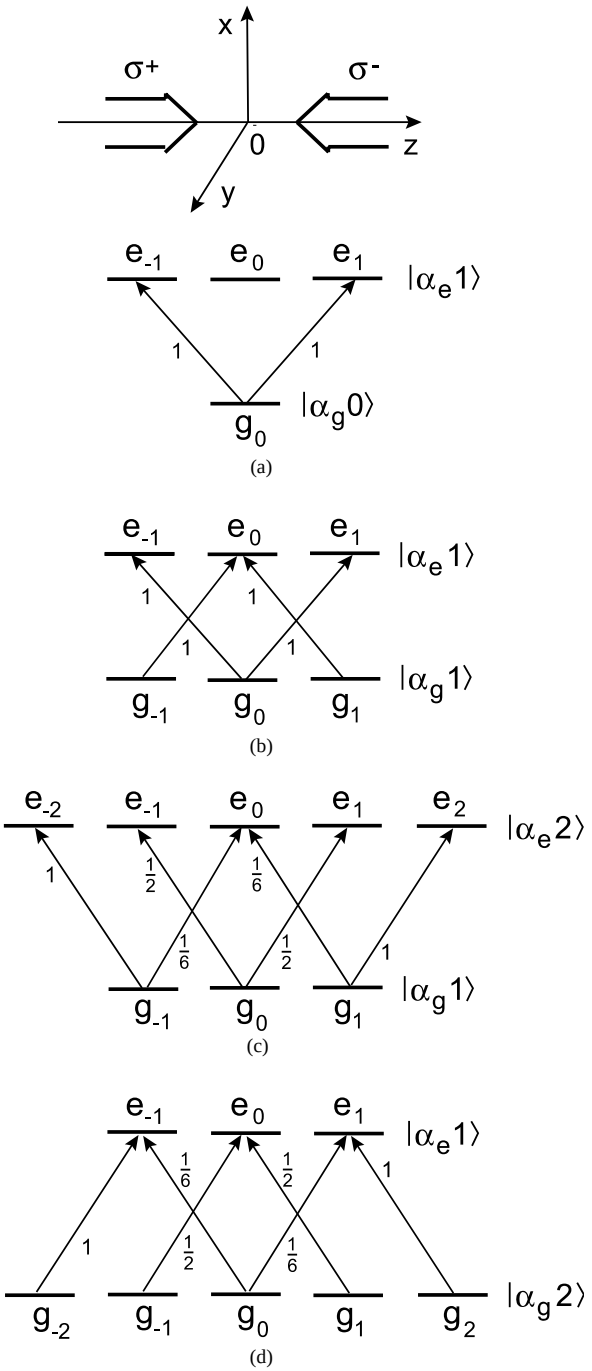

Fig. 3. Schemes of a (1+3)-level (a), (3+3)-level (b), (3+5)-level (c), and (5+3)-level atom (d) excited by counter-propagating circularly polarized laser waves composing a $\sigma^+ - \sigma^-$ field configuration. Arrows show the σ^+ ($g_M \rightarrow e_{M+1}$) and σ^- ($g_M \rightarrow e_{M-1}$) excitation transitions. Numbers show the relative strengths of the dipole $\sigma^{\pm}$-transitions.

3.3.1. (1+3)-level atom

One of the simplest multilevel schemes describing the interaction of a (1+3)-level atom with a $\sigma^+ - \sigma^-$ laser field configuration (82) is shown in Fig. 3a. The model refers to the atom with total momentum $F_g = 0$ in the ground state $|\alpha_g 0\rangle$ and total momentum $F_e = 1$ in the excited state $|\alpha_e 1\rangle$. For this scheme the basic types of the semiclassical density matrix equations (61) are as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} &= \mathrm{i}\Omega(\mathrm{e}^{\mathrm{i}kz}\rho_{e_{-1}g_0} + \mathrm{e}^{-\mathrm{i}kz}\rho_{e_1g_0})\mathrm{e}^{\mathrm{i}\delta t} + \text{c.c.} + 2\gamma(\rho_{e_{-1}e_{-1}} + \rho_{e_0e_0} + \rho_{e_1e_1}) \ , \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{e_{-1}e_{-1}} &= \mathrm{i}\Omega\mathrm{e}^{-\mathrm{i}kz-\mathrm{i}\delta t}\rho_{g_0e_{-1}} + \text{c.c.} - 2\gamma\rho_{e_{-1}e_{-1}} \ , \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0e_{-1}} &= -\mathrm{i}\Omega\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}(\rho_{g_0g_0} - \rho_{e_{-1}e_{-1}}) + \mathrm{i}\Omega\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho_{e_1e_{-1}} - \gamma\rho_{g_0e_{-1}} \ ,
\end{aligned} \tag{83}$$

where the Rabi frequency Ω is defined with respect to the transition between the nondegenerate states $|g_0\rangle$ and $|e_1\rangle$,

$$\Omega = \frac{\langle e_1|d_+|g_0\rangle E_0}{2\hbar} = \frac{\langle \alpha_e 1\|d\|\alpha_g 0\rangle E_0}{2\sqrt{3}\hbar} \ . \tag{84}$$

For long interaction times, $\tau_{\mathrm{int}} \gg \gamma^{-1}$, the model reduces to a V-type model since any initial population of the upper state $|e_0\rangle$ reduces to zero.

Excluding from Eqs. (83) the explicit time and coordinate dependence by evident substitutions,

$$\rho_{g_0e_{-1}} = \sigma_{g_0e_{-1}}\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}, \quad \rho_{g_0e_1} = \sigma_{g_0e_1}\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}, \quad \rho_{e_{-1}e_1} = \sigma_{e_{-1}e_1}\mathrm{e}^{-2\mathrm{i}kz} \ ,$$

and putting the total time derivatives equal to zero, one can find the steady-state values for the quantities σ_{kl} and find the dipole radiation force according to basic formula (65) as $\mathbf{F} = F\mathbf{e}_z$,

$$F = \langle \mathbf{d} \rangle \cdot \partial \mathbf{E}/\partial z = 2\hbar k\Omega \,\mathrm{Im}(\sigma_{g_0e_{-1}} - \sigma_{g_0e_1}) \ . \tag{85}$$

The velocity dependence of the dipole radiation force (85) can be understood from an approximate expression valid to a second order in a small Rabi frequency Ω or a first order in a small saturation parameter $G = 2\Omega^2/\gamma^2$,

$$F = \hbar k\,\gamma G(L_- - L_+) \ , \tag{86}$$

where $L_\pm = \gamma^2/(\gamma^2 + (\delta \pm kv)^2)$ are the Lorentz factors and $v = v_z$ is the velocity projection on Oz axis. Eq. (86) shows that the velocity dependence of the force comes from two one-photon absorption processes described by the Lorentz factors $L_\pm$ (Fig. 4). According to the resonance conditions, the one-photon absorption processes are located near resonant velocities $v_{\mathrm{res}} = \pm\delta/k$. At low optical saturation, the force is mainly defined by the terms which describe an independent excitation of the atom by two counter-propagating laser waves (82). At higher saturation the force includes additional terms which describe the cross-saturation of the two optical transitions sharing the common ground level $|g_0\rangle = |\alpha_g 0\rangle$. Note, that the condition of validity of force (85) coincides with condition (66) for a two-level atom.

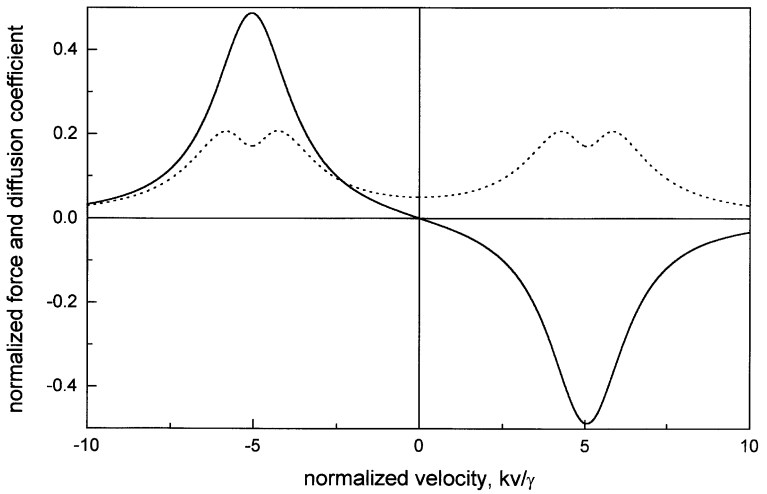

Fig. 4. Velocity dependence of the radiation force $F/\hbar k\gamma$ (solid line) and longitudinal diffusion coefficient $D_{zz}/\hbar^2k^2\gamma$ (dotted line) for a (1+3)-level atom in a $\sigma^+ - \sigma^-$ laser field configuration at detuning $\delta = -5\gamma$ and saturation parameter $G = 1$.

3.3.2. (3+3)-level atom

In the case of interaction of a (3+3)-level atom with a $\sigma^+ - \sigma^-$ laser field configuration (Fig. 3b) the structure of the semiclassical density matrix equations (61) can be seen from the example equations,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_{-1}} &= \mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho_{e_0g_{-1}} + \text{c.c.} + \gamma(\rho_{e_{-1}e_{-1}} + \rho_{e_0e_0}) \,, \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} &= \mathrm{i}\Omega(\mathrm{e}^{-\mathrm{i}\delta t}\rho_{g_{-1}e_0} + \mathrm{e}^{\mathrm{i}\delta t}\rho_{e_0g_1})\mathrm{e}^{-\mathrm{i}kz} - \gamma\rho_{e_{-1}e_1} \,, \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}e_0} &= -\mathrm{i}\Omega(\rho_{g_{-1}g_{-1}} - \rho_{e_0e_0})\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t} + \mathrm{i}\Omega\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho_{g_{-1}g_1} - \gamma\rho_{g_{-1}e_0} \,,
\end{aligned} \tag{87}$$

where the Rabi frequency Ω is defined with respect to the transition between the states $|g_0\rangle$ and $|e_1\rangle$,

$$\Omega = \frac{\langle e_1|d_+|g_0\rangle E_0}{2\hbar} = \frac{\langle \alpha_e 1\|d\|\alpha_g 1\rangle E_0}{2\sqrt{6}\hbar} \,. \tag{88}$$

For long times, $\tau_{\mathrm{int}} \gg \gamma^{-1}$, this interaction scheme reduces to a Λ-type scheme. This happens because the spontaneous decays are forbidden at the transition $|e_0\rangle - |g_0\rangle$ and, accordingly, the entire atomic population finally goes to three states, $|g_{-1}\rangle$, $|g_1\rangle$ and $|e_0\rangle$ (Fig. 5). At zero velocity $v_z = v = 0$ the population of the upper state $|e_0\rangle$ becomes equal to zero due to the coherent population trapping effect (Arimondo and Orriols, 1976; Gray et al., 1978).

The dipole radiation force for this scheme is equal to zero for all velocities. It can be shown from Eqs. (87) that zero value of the force is closely connected with the equality of the spontaneous decay rates at neighboring transitions $|e_0\rangle \to |g_{-1}\rangle$ and $|e_0\rangle \to |g_1\rangle$. When the decay rates in a

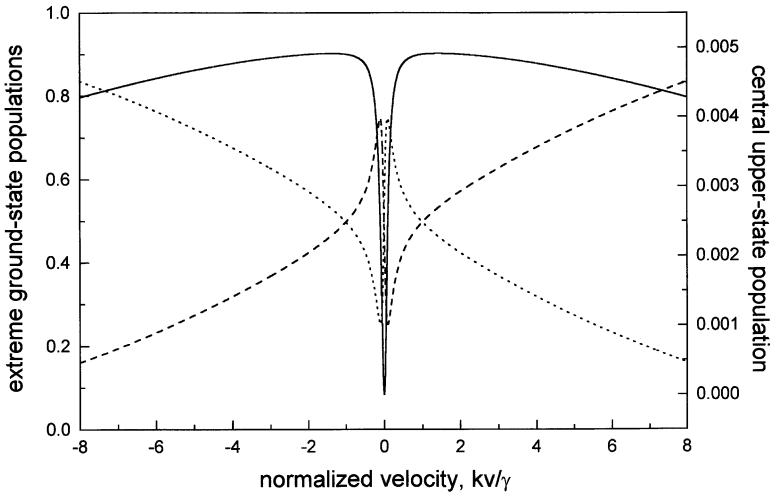

Fig. 5. Velocity dependences of the stationary ground-state extreme populations $N_{-1}=\rho_{g_{-1}g_{-1}}$ (dashed line, refers to left scale) and $N_1=\rho_{g_1g_1}$ (dotted line, refers to left scale) and upper-state central population $n_0=\rho_{e_0e_0}$ (solid line, refers to right scale) for a (3+3)-level atom in a $\sigma^+-\sigma^-$ laser field configuration at detuning $\delta=-20\gamma$ and saturation parameter $G=4$.

model of a Λ-type atom are chosen to be nonequal the force becomes different from zero. This last model is, however, artificial for the real atomic schemes.

3.3.3. (3+5)-level atom

A $(3+5)$-level interaction scheme shown in Fig. 3c represents a special interest since this scheme illustrates the contributions of the two-photon optical processes in the dipole radiation force on a simple example. Basic types of the semiclassical density matrix equations (61) for a (3+5)-level atom interacting with the field (82) can be described by the example equations,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} &= \frac{\mathrm{i}\Omega}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}kz}\rho_{e_{-1}g_0}+\mathrm{e}^{-\mathrm{i}kz}\rho_{e_1g_0})\mathrm{e}^{\mathrm{i}\delta t}+\mathrm{c.c.}+\gamma\left(\rho_{e_{-1}e_{-1}}+\frac{4}{3}\rho_{e_0e_0}+\rho_{e_1e_1}\right)\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} &= \mathrm{i}\Omega(\mathrm{e}^{\mathrm{i}\delta t}\rho_{e_{-2}g_1}-\mathrm{e}^{-\mathrm{i}\delta t}\rho_{g_{-1}e_2})\mathrm{e}^{\mathrm{i}kz}+\frac{\mathrm{i}\Omega}{\sqrt{6}}(\mathrm{e}^{\mathrm{i}\delta t}\rho_{e_0g_1}-\mathrm{e}^{-\mathrm{i}\delta t}\rho_{g_{-1}e_0})\mathrm{e}^{-\mathrm{i}kz}\\
&\quad+\gamma\left(\sqrt{\frac{2}{3}}\rho_{e_{-2}e_0}+\rho_{e_{-1}e_1}+\sqrt{\frac{2}{3}}\rho_{e_0e_2}\right)\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_1e_2} &= -\mathrm{i}\Omega(\rho_{g_1g_1}-\rho_{e_2e_2})\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}+\frac{\mathrm{i}\Omega}{\sqrt{6}}\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho_{e_0e_2}-\gamma\rho_{g_1e_2}\,,
\end{aligned}\tag{89}$$

where the Rabi frequency Ω is defined with respect to the most strong dipole transition in the scheme of Fig. 3c,

$$\Omega=\frac{\langle e_2|d_+|g_1\rangle E_0}{2\hbar}=\frac{\langle\alpha_e 2\|d\|\alpha_g 1\rangle E_0}{2\sqrt{5}\hbar}\,.\tag{90}$$

For this interaction scheme the explicit time and position dependence can be excluded from the equations with obvious substitutions:

$$\rho_{g_{-1}g_1} = \sigma_{g_{-1}g_1} e^{-2ikz}, \quad \rho_{g_1e_2} = \sigma_{g_1e_2} e^{-ikz+i\delta t}, \dots \ . \tag{91}$$

After that the density matrix equations can be solved for a steady-state case.

The dipole radiation force $\mathbf{F} = F\mathbf{e}_z$ on a (3+5)-level atom is determined by the steady-state density matrix elements according to Eq. (65),

$$\begin{aligned} F &= \langle \mathbf{d} \rangle \cdot \partial \mathbf{E}/\partial z \\ &= 2\hbar k \Omega \, \mathrm{Im} \left[(\sigma_{g_{-1}e_{-2}} - \sigma_{g_1e_2}) + \frac{1}{\sqrt{2}} (\sigma_{g_0e_{-1}} - \sigma_{g_0e_1}) + \frac{1}{\sqrt{6}} (\sigma_{g_1e_0} - \sigma_{g_{-1}e_0}) \right] \ . \end{aligned} \tag{92}$$

Before discussing the dipole radiation force (92) it is worth noting that the conditions of validity of the force for a scheme possessing the ground-state sublevels are much stronger than that (66) for a two-level or a (1+3)-level atom. While the atomic coherence induced by the one-photon processes decays to a steady state at the spontaneous decay rate γ, the ground-state coherence $\sigma_{g_{-1}g_1}$ decays to a steady state at the rates of the one-photon processes,

$$W_2^{\pm} = \frac{\Omega^2}{(\gamma^2 + \delta_{\pm}^2)} \gamma \ , \tag{93}$$

where $\delta_{\pm} = \delta \pm kv$ and $v = v_z$. At large detunings and small velocities a relatively slow decay of the ground-state coherence imposes in addition to condition (66) a relatively strong condition on the interaction time,

$$\tau_{\mathrm{int}} \gg \left(\frac{\delta}{\Omega} \right)^2 \gamma^{-1} \ . \tag{94}$$

The structure of force (92) under conditions (66) and (94) can be clearly seen from the expression valid at a low optical saturation, i.e. at small effective saturation parameters $s_{\pm}$,

$$s_{\pm} = \Omega^2/(\gamma^2 + \delta_{\pm}^2) \ll 1 \ , \tag{95}$$

In this case the radiation force can be shown to be (Chang et al., 1999b):

$$\begin{aligned} F = \hbar k \gamma \, [&(s_- - s_+) N_0 - (2s_+ - s_-/3) N_- \\ &+ \frac{1}{3} (s_- - s_+) \, \mathrm{Re} \, v + \frac{1}{3\gamma} (s_- \delta_- + s_+ \delta_+) \, \mathrm{Im} \, v \Big] \ , \end{aligned} \tag{96}$$

where $N_- = \rho_{g_{-1}g_{-1}}$, $N_0 = \rho_{g_0g_0}$, and $N_+ = \rho_{g_1g_1}$ are the populations of the ground-state sublevels and $v = \sigma_{g_{-1}g_1}$ is the ground-state coherence.

An analytical solution to steady-state equations for the density matrix elements under conditions (66) and (95) gives the ground-state populations and coherence as

$$N_- = \frac{1}{2\tilde{\Delta}} \left(9 + \frac{3}{5} \frac{\mu^2}{k^2v^2 + \mu^2} \left(\frac{13}{5} \frac{\delta^2}{\gamma^2} - 2 \right) - \frac{3\mu k v}{k^2v^2 + \mu^2} \frac{\delta}{\gamma} \right) \ ,$$

$$N_0 = \frac{2}{\tilde{\Delta}} \left(1 + \frac{1}{5} \frac{\mu^2}{k^2v^2 + \mu^2} \left(\frac{6}{5} \frac{\delta^2}{\gamma^2} + 1 \right) \right) \ ,$$

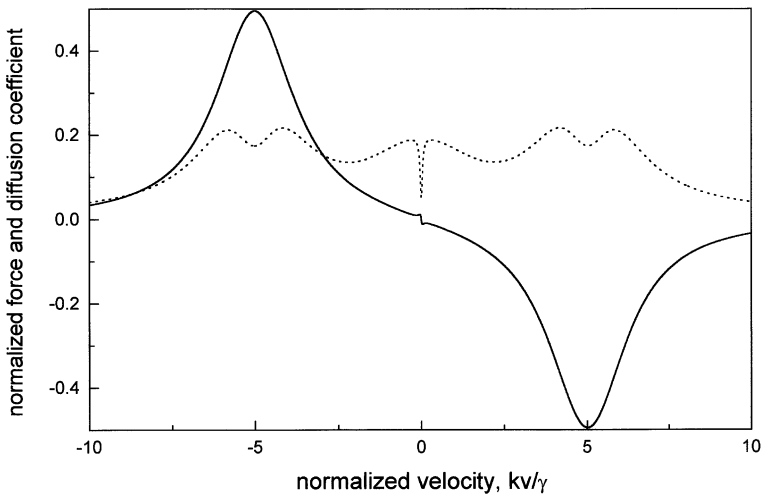

Fig. 6. Dipole radiation force $F/\hbar k\gamma$ (solid line) and longitudinal diffusion coefficient $D_{zz}/\hbar^2k^2\gamma$ (dotted line) for a (3+5)-level atom in a $\sigma^+ - \sigma^-$ field configuration as functions of velocity $v = v_z$ for detuning $\delta = -5\gamma$ and saturation parameter $G = 1$.

$$N_+ = \frac{1}{2\tilde{\Delta}}\left(9 + \frac{3}{5}\frac{\mu^2}{k^2v^2+\mu^2}\left(\frac{13}{5}\frac{\delta^2}{\gamma^2} - 2\right) + \frac{3\mu kv}{k^2v^2+\mu^2}\frac{\delta}{\gamma}\right) ,$$

$$v = \frac{3}{2\tilde{\Delta}}\frac{\mu}{k^2v^2+\mu^2}\left(\left(1 + \frac{1}{5}\frac{\delta^2}{\gamma^2}\right)\mu + \mathrm{i}kv\right) , \tag{97}$$

where the common denominator is

$$\tilde{\Delta} = 11 + \frac{1}{5}\frac{\mu^2}{k^2v^2+\mu^2}\left(\frac{51}{5}\frac{\delta^2}{\gamma^2} - 4\right) . \tag{98}$$

The last equations show that the force includes the terms coming from both one-photon absorption (emission) processes and two-photon processes possessing the frequency width

$$\mu = \frac{5}{12}\frac{G\gamma}{1+\delta^2/\gamma^2} , \tag{99}$$

where G is a dimensionless saturation parameter,

$$G = \frac{2\Omega^2}{\gamma^2} = \frac{1}{10}\left(\frac{\langle\alpha_e 2\|d\|\alpha_g 1\rangle E_0}{\hbar\gamma}\right)^2 . \tag{100}$$

According to Eqs. (97) the two-photon processes are located at zero velocity (Fig. 6). The substitution of the ground-state populations and ground-state coherence found in a lowest order in small parameters $s_\pm$ into Eq. (96) gives an explicit expression for the force valid in a low-saturation limit. For a low-velocity region, $kv \ll \gamma$, the force (96) is (Dalibard and Cohen-Tannoudji, 1989;

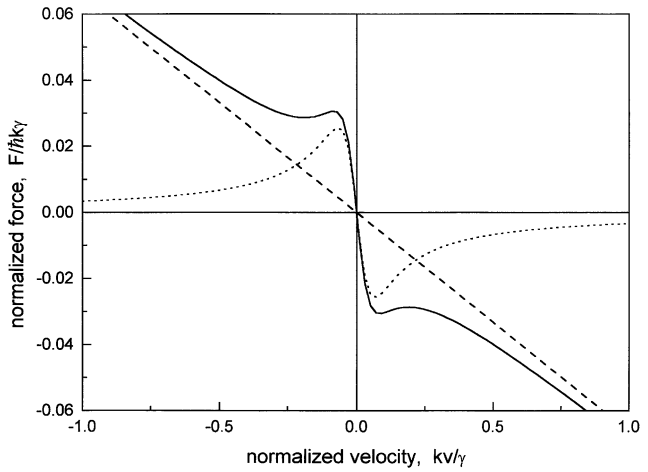

Fig. 7. Radiation force F (solid line), the incoherent part $F_{\rm in}$ of the force (dashed line) and coherent part $F_{\rm c}$ of the force (dotted line) for a (3+5)-level atom interacting with a $\sigma^+ - \sigma^-$ field configuration as functions of velocity $v = v_z$ for the detuning $\delta = -3\gamma$ and saturation parameter $G = 1$.

Chang et al., 1999a, b):

$$F = \frac{25}{11}\hbar k\gamma \frac{G}{(1+\delta^2/\gamma^2)^2}\frac{k^2v^2 + (88/85)\tilde{\mu}^2}{k^2v^2+\tilde{\mu}^2}\frac{\delta kv}{\gamma^2} + \frac{5}{44}\hbar k\gamma \frac{G^2}{(1+\delta^2/\gamma^2)^2}\frac{\delta kv}{k^2v^2+\tilde{\mu}^2}\,, \tag{101}$$

where $\tilde{\mu}$ is an effective halfwidth of the two-photon resonance perturbed by the one-photon processes,

$$\tilde{\mu} = \sqrt{\frac{17}{33}}\frac{G\gamma\sqrt{5+\delta^2/\gamma^2}}{4(1+\delta^2/\gamma^2)}\,. \tag{102}$$

Eq. (101) directly shows that in a low-velocity region the force includes two different parts (Fig. 7). The first part of force (101) comes mainly due to the contribution of the incoherent one-photon absorption (emission) processes slightly perturbed by the two-photon coherent processes. This incoherent part of the force, $F_{\rm in}$, has the same physical origin as the force on a $(1+3)$-level atom in the field of two counter-propagating waves (see Eq. (86)). At large detunings, $|\delta| \gtrsim \gamma \gg \tilde{\mu}$, the velocity dependence of the incoherent part of the force is a broad one since the one-photon absorption (emission) resonances are centered at the resonance velocities $kv_{\rm res} = \pm\delta$. The second coherent part of the force, $F_{\rm c}$, that includes the dependence on the square of the laser field intensity, is due to the two-photon resonance processes broadened by the one-photon processes. This part of the force is located near zero velocity in the velocity region $|v| \lesssim \tilde{\mu}/k$. Other functions describing the internal state of the atom show similar behavior. In particular, two photon processes perturbed by single-photon absorption (emission) processes are responsible for sharp variations of the ground-state populations near zero velocity (Fig. 8).

Qualitative behavior of force (92) and atomic populations at low velocities can be described by simple physical arguments. The velocity position of the narrow two-photon structures can be estimated from the energy conservation law. In the atom rest frame the absorption of a photon from one travelling wave and the emission of a photon into the other travelling wave results in a two-photon transition between the ground-state sublevels $|g_{-1}\rangle$, $|g_{+1}\rangle$ that does not change the

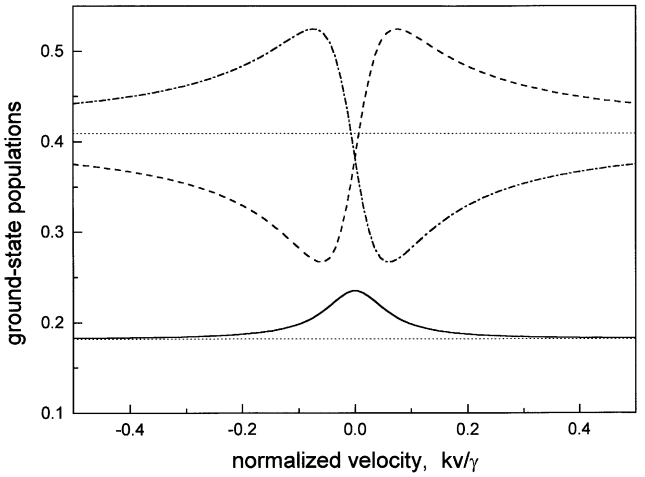

Fig. 8. Ground-state populations $N_- = \rho_{g_{-1}g_{-1}}$, (dashed line), $N_0 = \rho_{g_0g_0}$ (solid line), and $N_+ = \rho_{g_1g_1}$ (dash-dotted line) for a (3+5)-level atom in a $\sigma^+ - \sigma^-$ field configuration as functions of velocity $v = v_z$ for detuning $\delta = -3\gamma$ and saturation parameter $G = 1$. The dotted lines show the values of the ground-state populations in a limit of a zero saturation when $N_0 = 4/22$ and $N_\pm = 9/22$.

atom energy, $(\omega \pm kv) - (\omega \mp kv) \approx 0$. The energy conservation law thus shows that two-photon resonance structure in the force is located at zero velocity, $kv \approx 0$. The widths of the narrow resonance structures can also be estimated from a simple physical argument. For the atom not perturbed by any external interaction, the decay rate for the ground-state coherence is zero. The laser field connects the ground-state probability amplitudes with upper-state probability amplitudes through the dipole interaction term $\hbar\Omega$ and accordingly causes the decay rate of the ground-state coherence to be of the order of the rate of dipole transitions, i.e. of the order of $\gamma\Omega^2/(\gamma^2 + \delta^2)$. This quantity accordingly plays the role of the frequency width μ for two-photon resonance processes as determined by Eq. (99). When the two-photon processes are perturbed by the one-photon processes the two-photon structures become broader thus obtaining the final width $\tilde{\mu}$, which at large detuning is defined by the light shift of the ground-state sublevels, $\tilde{\mu} \approx \Omega^2/|\delta|$.

Some other mathematical approaches to the analysis of the above model including computer simulations can be found in Mølmer, 1991.

3.3.4. (5+3)-level atom

For a $(5+3)$-level atom interacting with the field (82) the semiclassical density matrix equations (61) are illustrated by some basic examples as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} &= \frac{\mathrm{i}\Omega}{\sqrt{6}}(\mathrm{e}^{\mathrm{i}kz}\rho_{e_{-1}g_0} + \mathrm{e}^{-\mathrm{i}kz}\rho_{e_1g_0})\mathrm{e}^{\mathrm{i}\delta t} + \mathrm{c.c.} + \gamma\left(\frac{1}{5}\rho_{e_{-1}e_{-1}} + \frac{4}{5}\rho_{e_0e_0} + \frac{1}{5}\rho_{e_1e_1}\right), \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} &= \frac{\mathrm{i}\Omega}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\delta t}\rho_{e_0g_1} - \mathrm{e}^{-\mathrm{i}\delta t}\rho_{g_{-1}e_0})\mathrm{e}^{-\mathrm{i}kz} + \frac{3}{5}\gamma\rho_{e_{-1}e_1}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{e_1g_0} &= \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho_{g_0g_0} - \rho_{e_1e_1})\mathrm{e}^{\mathrm{i}kz - \mathrm{i}\delta t} + \mathrm{i}\Omega\left(\rho_{g_2g_0} - \frac{1}{\sqrt{6}}\rho_{e_1e_{-1}}\right)\mathrm{e}^{-\mathrm{i}kz - \mathrm{i}\delta t} - \gamma\rho_{e_1g_0},
\end{aligned} \tag{103}$$

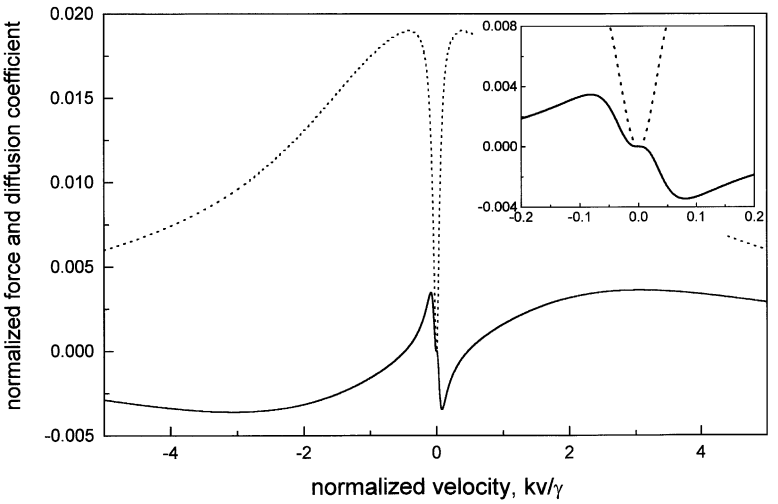

Fig. 9. The dipole radiation force $F/\hbar k\gamma$ (solid line) and longitudinal diffusion coefficient $D_{zz}/\hbar^2k^2\gamma$ (dotted line) for a (5+3)-level atom in a $\sigma^+ - \sigma^-$ field configuration as functions of velocity $v = v_z$ for detuning $\delta = -5\gamma$ and saturation parameter $G = 1$. The inset shows the structure of the force and the diffusion coefficient in a small velocity region.

where the Rabi frequency Ω is defined with respect to the most strong dipole transition in the scheme of Fig. 3d,

$$\Omega = \frac{\langle e_{-1}|d_+|g_{-2}\rangle E_0}{2\hbar} = \frac{\langle \alpha_e 1\|d\|\alpha_g 2\rangle E_0}{2\sqrt{5}\hbar} \ . \tag{104}$$

For this scheme the dipole radiation force consists of two usual broad resonance structures located at resonance velocities $v_{\rm res} = \pm\delta/k$ and additional narrow and supernarrow structures near zero velocity (Fig. 9). The narrow structure in the force reflects the coherent population trapping effect in a Λ-scheme presented by the magnetic sublevels $|g_{-1}\rangle$, $|g_1\rangle$, and $|e_0\rangle$. The contribution of the coherent population trapping effect can be clearly seen in the velocity dependence of the upper-state populations (Fig. 10). Supernarrow structure comes from the four-photon processes in the M-type scheme presented by the sublevels $|g_{-2}\rangle$, $|g_0\rangle$, $|g_2\rangle$, and $|e_{-1}\rangle$, $|e_1\rangle$. This supernarrow structure is responsible for zero slope of the force at zero velocity. The conditions of validity for the force on a $(5+3)$-level atom are similar to that for a $(3+5)$-level atom.

3.4. Multilevel atoms in lin⊥lin field configuration

The other important case of a dipole interaction of the multilevel atoms with the laser field is the case of interaction with the field composed of two counter-propagating waves linearly polarized along orthogonal directions (Fig. 11a). The electric field of this kind known as a lin⊥lin configuration can be represented as

$$\mathbf{E} = \mathbf{e}_x E_0 \cos(kz - \omega t) + \mathbf{e}_y E_0 \cos(kz + \omega t) \ . \tag{105}$$

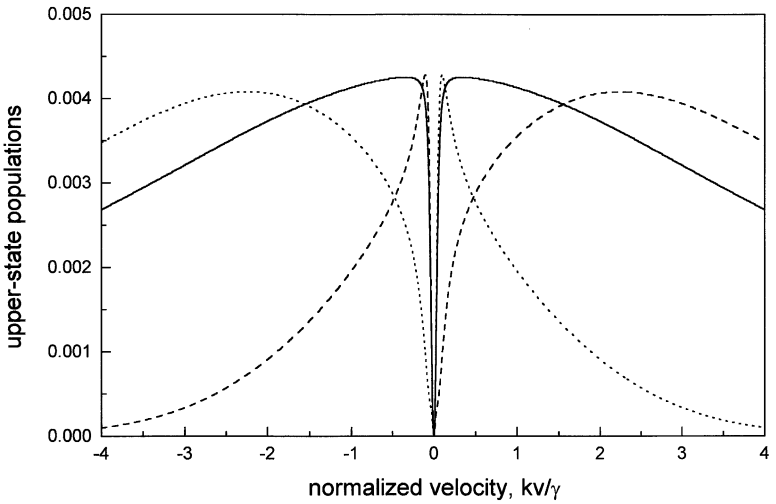

Fig. 10. Stationary upper-state populations $n_0 = \rho_{e_0e_0}$ (solid line), $n_{-1} = \rho_{e_{-1}e_{-1}}$ (dashed line), and $n_1 = \rho_{e_1e_1}$ (dotted line) for a (5+3)-level atom in a $\sigma^+ - \sigma^-$ laser field configuration at detuning $\delta = -5\gamma$ and saturation parameter $G = 1$.

By decomposing the Cartesian unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ over the spherical unit vectors (20) one can equivalently represent the electric field (105) as a superposition of the fields of two $\sigma^+ - \sigma^-$ configurations (Fig. 11b),

$$\begin{aligned}
&\mathbf{E} = \mathbf{E}_{1\sigma} + \mathbf{E}_{2\sigma} \,, \\
&\mathbf{E}_{1\sigma} = \frac{E_0}{2\sqrt{2}} \left[(-\mathbf{e}_+ \mathrm{e}^{\mathrm{i}(kz-\omega t)} + \mathbf{e}_- \mathrm{e}^{-\mathrm{i}(kz-\omega t)}) + \mathrm{i}(\mathbf{e}_+ \mathrm{e}^{\mathrm{i}(kz+\omega t)} + \mathbf{e}_- \mathrm{e}^{-\mathrm{i}(kz+\omega t)})\right] , \\
&\mathbf{E}_{2\sigma} = \frac{E_0}{2\sqrt{2}} [\mathrm{i}(\mathbf{e}_+ \mathrm{e}^{-\mathrm{i}(kz+\omega t)} + \mathbf{e}_- \mathrm{e}^{\mathrm{i}(kz+\omega t)}) + (-\mathbf{e}_+ \mathrm{e}^{-\mathrm{i}(kz-\omega t)} + \mathbf{e}_- \mathrm{e}^{\mathrm{i}(kz-\omega t)})] \,.
\end{aligned} \tag{106}$$

In Eqs. (106) the first $\sigma^+ - \sigma^-$ configuration includes a σ^+-polarized wave propagating along Oz axis in the positive direction and a σ^--polarized wave propagating in the negative direction. The second $\sigma^+ - \sigma^-$ configuration includes a σ^+-polarized wave propagating along Oz axis in the negative direction and a σ^--polarized wave propagating in the positive direction (Fig. 11b). At the origin of the reference frame, $z = 0$, the fields $\mathbf{E}_{1\sigma}$ and $\mathbf{E}_{2\sigma}$ are shifted in phase $\pi/2$.

Below two examples of the multilevel dipole interaction schemes which include the field in the form of a lin⊥lin configuration, a $(3 + 3)$-level scheme and a $(3 + 5)$-level scheme are considered.

3.4.1. (3+3)-level atom

In the case of interaction of a $(3 + 3)$-level atom with a lin⊥lin field configuration (106) (Fig. 11c) the semiclassical density matrix equations (61) written in RWA are similar to Eqs. (87). Some basic example equations for this scheme are

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho_{g_{-1}g_{-1}} = \Omega(\mathrm{e}^{\mathrm{i}kz} - \mathrm{i}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} \rho_{e_0g_{-1}} + \mathrm{c.c.} + \gamma(\rho_{e_{-1}e_{-1}} + \rho_{e_0e_0}) \,,$$

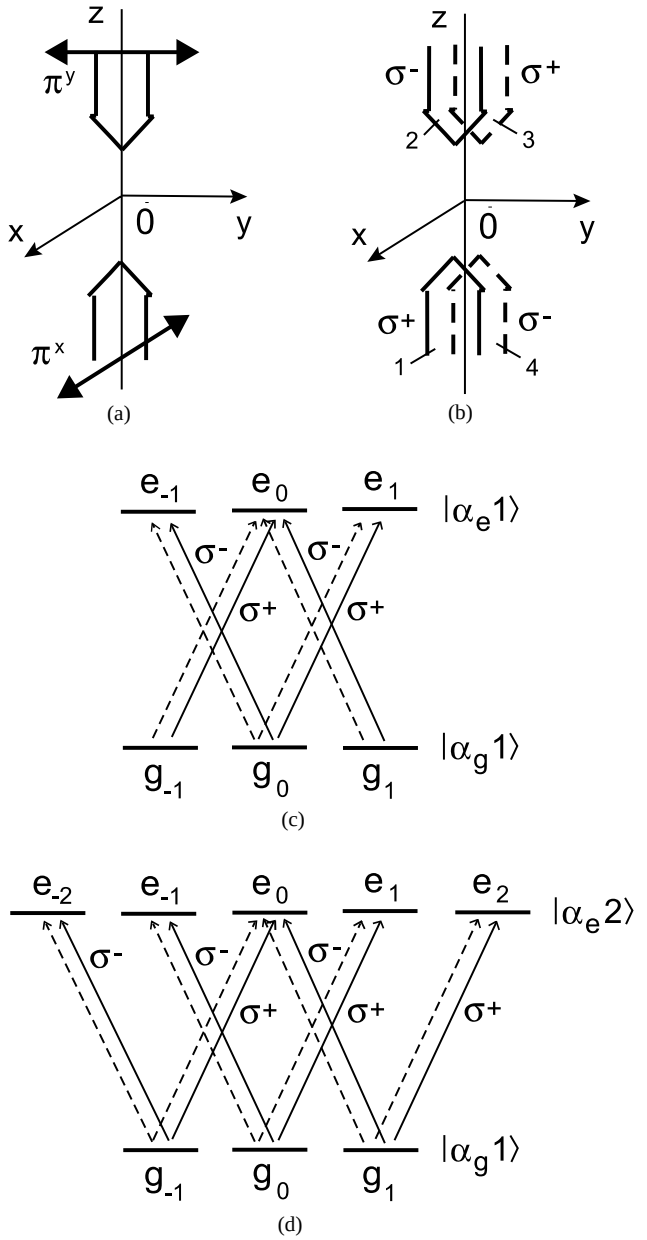

Fig. 11. Two counter-propagating linearly polarized laser waves comprising a lin⊥lin field configuration (a), decomposition of linearly polarized waves over four circularly polarized waves (b), scheme of transitions in a (3+3)-level atom interacting with a lin⊥lin field configuration (c), and scheme of transitions in a (3+5)-level atom interacting with a lin⊥lin field configuration (d).

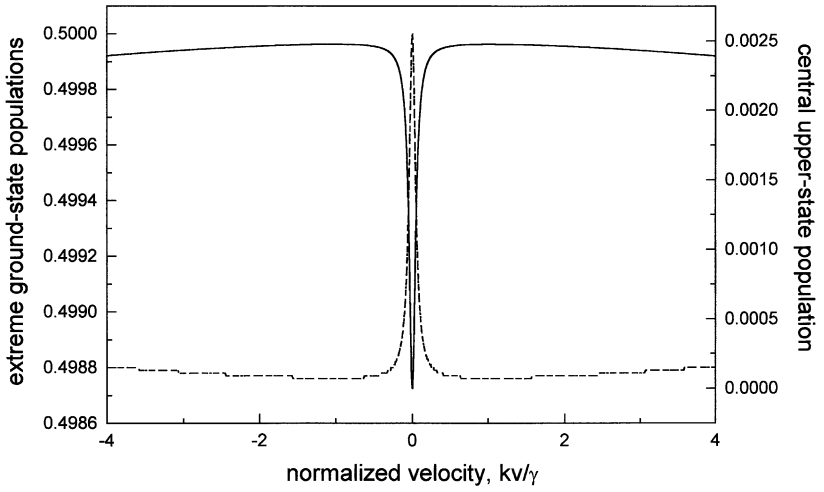

Fig. 12. Velocity dependences of the stationary ground-state extreme populations $N_{-1} = \rho_{g_{-1}g_{-1}}$ (dashed line, refers to left scale) and $N_1 = \rho_{g_1g_1}$ (dotted line, refers to left scale) and upper-state central population $n_0 = \rho_{e_0e_0}$ (solid line, refers to right scale) for a (3+3)-level atom in a lin⊥lin laser field configuration at detuning $\delta = -20\gamma$ and saturation parameter $G = 4$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \Omega(\mathrm{e}^{\mathrm{i}kz} - \mathrm{i}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t}\rho_{e_0g_1} - \Omega(\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{-\mathrm{i}\delta t}\rho_{g_{-1}e_0} - \gamma\rho_{e_{-1}e_1} \ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}e_0} = \Omega(\mathrm{e}^{\mathrm{i}kz} - \mathrm{i}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t}(\rho_{e_0e_0} - \rho_{g_{-1}g_{-1}}) + \Omega(\mathrm{e}^{\mathrm{i}kz} + \mathrm{i}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t}\rho_{g_{-1}g_1} - \gamma\rho_{g_{-1}e_0} \ ,$$

where the Rabi frequency Ω is defined by an equation similar to Eq. (88),

$$\Omega = \frac{\langle \alpha_e 1 \| d \| \alpha_g 1 \rangle E_0}{2\sqrt{12}\hbar} \ .$$

For long interaction times, $\tau_{\mathrm{int}} \gg \gamma^{-1}$, this scheme reduces to a Λ-type scheme which exhibits the coherent population trapping effect at zero velocity (Fig. 12). The dipole radiation force for this scheme is equal to zero for any velocities due to the same reason as in the case of a $\sigma^+ - \sigma^-$ configuration (see Section 3.3.2).

3.4.2. (3+5)-level atom

For the case of interaction of a $(3 + 5)$-level atom with a lin⊥lin configuration (Fig. 11d) basic types of the semiclassical atomic density matrix equations (61) are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} = \frac{\mathrm{i}\Omega}{\sqrt{2}}(\rho_{e_{-1}g_0}\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\rho_{e_{-1}g_0}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} - \frac{\mathrm{i}\Omega}{\sqrt{2}}(\rho_{e_1g_0}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho_{e_1g_0}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} + \mathrm{c.c.}$$
$$+ \gamma\left(\rho_{e_{-1}e_{-1}} + \frac{4}{3}\rho_{e_0e_0} + \rho_{e_1e_1}\right) \ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \mathrm{i}\Omega(\rho_{e_{-2}g_1}\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\rho_{e_{-2}g_1}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} - \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho_{e_0g_1}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho_{e_0g_1}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t}$$
$$- \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho_{g_{-1}e_0}\mathrm{e}^{\mathrm{i}kz} + \mathrm{i}\rho_{g_{-1}e_0}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{-\mathrm{i}\delta t} + \mathrm{i}\Omega(\rho_{g_{-1}e_2}\mathrm{e}^{\mathrm{i}kz} - \mathrm{i}\rho_{g_{-1}e_2}\mathrm{e}^{-\mathrm{i}kz})\mathrm{e}^{-\mathrm{i}\delta t}$$
$$+ \gamma\left(\sqrt{\frac{2}{3}}\rho_{e_{-2}e_0} + \rho_{e_{-1}e_1} + \sqrt{\frac{2}{3}}\rho_{e_0e_2}\right) ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_1e_2} = \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho_{e_0e_2}\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\rho_{e_0e_2}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} - \mathrm{i}\Omega(\rho_{e_2e_2}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho_{e_2e_2}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t}$$
$$+ \mathrm{i}\Omega(\rho_{g_1g_1}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho_{g_1g_1}\mathrm{e}^{\mathrm{i}kz})\mathrm{e}^{\mathrm{i}\delta t} - \gamma\rho_{g_1e_2} ,$$

where the Rabi frequency is defined as

$$\Omega = \frac{\langle e_2|d_+|g_1\rangle E_0}{2\sqrt{2}\hbar} = \frac{\langle \alpha_e 2\|d\|\alpha_g 1\rangle E_0}{2\sqrt{10}\hbar} . \tag{107}$$

In a set of equations describing the interaction of a $(3+5)$-level atom with a lin⊥lin configuration only an explicit time dependence can be excluded by simple substitutions for the off-diagonal elements,

$$\rho_{g_\alpha e_\beta} = \sigma_{g_\alpha e_\beta}\mathrm{e}^{\mathrm{i}\delta t} , \tag{108}$$

while the elimination of an explicit position dependence can be done by decomposing the density matrix elements into infinite series,

$$\begin{aligned}
\rho_{g_\alpha g_\beta} &= R^0_{g_\alpha g_\beta} + R^{-2}_{g_\alpha g_\beta}\mathrm{e}^{-2\mathrm{i}kz} + R^2_{g_\alpha g_\beta}\mathrm{e}^{2\mathrm{i}kz} + \cdots ,\\
\rho_{e_\alpha e_\beta} &= R^0_{e_\alpha e_\beta} + R^{-2}_{e_\alpha e_\beta}\mathrm{e}^{-2\mathrm{i}kz} + R^2_{e_\alpha e_\beta}\mathrm{e}^{2\mathrm{i}kz} + \cdots ,\\
\sigma_{g_\alpha e_\beta} &= S^{-1}_{g_\alpha e_\beta}\mathrm{e}^{-\mathrm{i}kz} + S^1_{g_\alpha e_\beta}\mathrm{e}^{\mathrm{i}kz} + \cdots .
\end{aligned} \tag{109}$$

The above decompositions show that the multiphoton processes play an important role in the interaction of a $(3+5)$-level atom with a lin⊥lin configuration. Identification of the processes follows most clearly from the equations considered to different orders of the rate equation approximation (REA). As usual, calculation of atomic functions to the $2n$th order REA implies that the ground and upper-state populations and coherences are calculated to $2n$th order while the optical coherences to $(2n-1)$th order. Considering atomic populations and coherences to second order REA, one takes into account the direct one-photon and two-photon processes and stepwise processes composed of the direct processes. When atomic functions are considered to fourth order REA one takes into account the direct one-, two-, three- and four-photon processes and stepwise processes composed of the above direct processes.

The multiphoton processes considerably modify the optical coherences and accordingly the dipole radiation force which is defined according to Eqs. (65) and (109) by a series representation

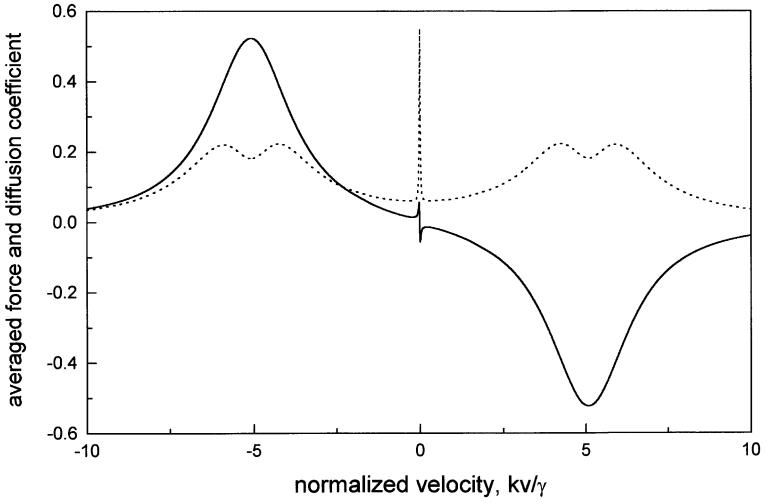

Fig. 13. Spatially averaged dipole radiation force $F^0/\hbar k\gamma$ (solid line) and longitudinal diffusion coefficient $D^0_{zz}/\hbar^2k^2\gamma$ (dotted line) for a (3+5)-level atom in a lin⊥lin field configuration as functions of velocity $v = v_z$ for detuning $\delta = -5\gamma$ and saturation parameter $G = 1$.

(Chang et al., 2000),

$$
\begin{aligned}
F &= \hbar k\Omega \sum_{n=0,\pm 2,\ldots} f_n \mathrm{e}^{\mathrm{i}nkz} \,, \\
f_n &= S^{n-1}_{e_2g_1} + S^{n-1}_{e_{-2}g_{-1}} - \mathrm{i}S^{n-1}_{g_1e_2} + \mathrm{i}S^{n-1}_{g_{-1}e_{-2}} + S^{n+1}_{g_1e_2} + S^{n+1}_{g_{-1}e_{-2}} + \mathrm{i}S^{n+1}_{e_2g_1} - \mathrm{i}S^{n+1}_{e_{-2}g_{-1}} \\
&\quad + \frac{1}{\sqrt{6}}\left(S^{n-1}_{e_0g_1} + S^{n-1}_{e_0g_{-1}} - \mathrm{i}S^{n-1}_{g_{-1}e_0} + \mathrm{i}S^{n-1}_{g_1e_0} + S^{n+1}_{g_1e_0} + S^{n+1}_{g_{-1}e_0} + \mathrm{i}S^{n+1}_{e_0g_{-1}} - \mathrm{i}S^{n+1}_{e_0g_1}\right) \\
&\quad + \frac{1}{\sqrt{2}}\left(S^{n-1}_{e_1g_0} + S^{n-1}_{e_{-1}g_0} - \mathrm{i}S^{n-1}_{g_0e_1} + \mathrm{i}S^{n-1}_{g_0e_{-1}} + S^{n+1}_{g_0e_1} + S^{n+1}_{g_0e_{-1}} + \mathrm{i}S^{n+1}_{e_1g_0} - \mathrm{i}S^{n+1}_{e_{-1}g_0}\right) ,
\end{aligned}
\tag{110}
$$

where the harmonics of the force satisfy the "hermiticity" conditions, $f^*_n = f_{-n}$. The velocity dependence of a zero harmonic of the radiation force (110) shown in Fig. 13 clearly exhibits a narrow multiphoton structure at zero velocity.

Multiphoton processes specific to a lin⊥lin configuration also considerably modify the velocity dependence of any atomic function as compared with the case of a $\sigma^+ - \sigma^-$ configuration. As an example Fig. 14 shows zero harmonics of the ground-state populations $R^0_- = R^0_{g_{-1}g_{-1}}$, $R^0_0 = R^0_{g_0g_0}$ and $R^0_+ = R^0_{g_1g_1}$ for a (3+5)-level atom in a lin⊥lin configuration as functions of atomic velocity. For comparison, dashed lines in Fig. 14 show the ground-state populations $N_0 = \rho_{g_0g_0}$, $N_\pm = \rho_{g_{\pm1}g_{\pm1}}$ for the case when the atom is excited by the first $\sigma^+ - \sigma^-$ field configuration, $\mathbf{E}_{1\sigma}$, defined by Eq. (106). Note that in the last case the density matrix elements ρ_{kl} are defined by the equations which differ from Eqs. (89) by the interchange of the field (82) by the field $\mathbf{E}_{1\sigma}$. As can be seen from Fig. 14, atomic populations for a lin⊥lin field configuration include narrow resonance structures of two types, narrow and super-narrow structures located at zero velocity. Narrow structures are similar to

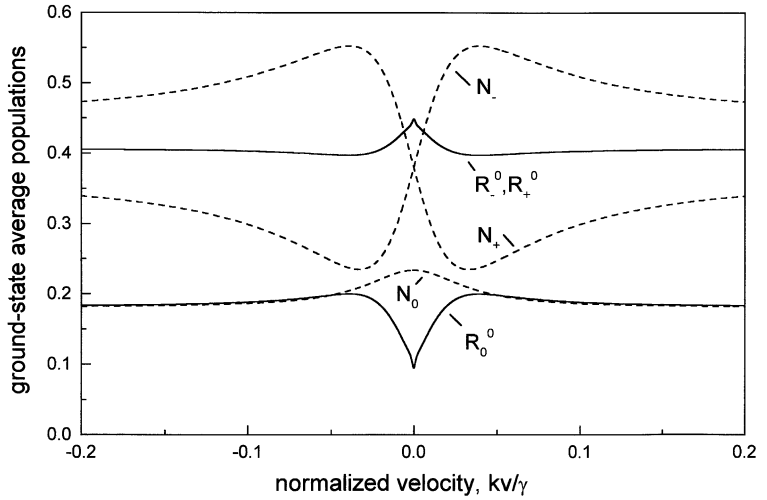

Fig. 14. Spatially averaged ground-state atomic populations R^0_-, R^0_0 and R^0_+ for a (3+5)-level atom in a lin⊥lin field configuration (solid lines) and ground-state populations N_-, N_0 and N_+ for a (3+5)-level atom in the first $\sigma^+ - \sigma^-$ configuration (dashed lines) as functions of velocity $v = v_z$ for saturation parameter $G = 2$ and detuning $\delta = -20\gamma$.

those for the case of a $\sigma^+ - \sigma^-$ configuration while super-narrow structures are features specific to a lin⊥lin configuration.

Solutions to the equations to different orders in series expansion (109) show that the super-narrow velocity structures appear in atomic populations already in second order REA while narrow velocity structures appear in fourth order REA only. The absence of the narrow structures in the atomic populations derived to second order REA shows that direct two-photon processes caused separately by two $\sigma^+ - \sigma^-$ configurations cancel each other. The super-narrow structures presented by second order REA are thus caused by the stepwise four-order process composed of the two-photon processes. In the fourth order REA a large number of the even-order processes jointly produce the narrow structures similar to that for a single $\sigma^+ - \sigma^-$ configuration.

It is instructive to compare the structure of the spatially averaged force for a lin⊥lin configuration with the force for a single $\sigma^+ - \sigma^-$ configuration. In a low-velocity region and at large detunings the spatially averaged force (110) for a lin⊥lin configuration (105) can be expressed through the ground-state populations and coherences as

$$F^0 = 2\hbar k\gamma \frac{\Omega^2}{\delta^2} \operatorname{Re}\left[\frac{5\delta}{3\gamma}(R^{+2}_+ - R^{+2}_-) - \frac{2\delta}{3\gamma}(v^2 - v^{-2}) - \frac{4}{3}v^0\right], \tag{111}$$

where $R^n_\pm = R^n_{g_{\pm1}g_{\pm1}}$ and $v^n = R^n_{g_{-1}g_1}$. In the same low-velocity large-detuning approximation the force for a single $\sigma^+ - \sigma^-$ configuration (82) is (Chang et al., 1999b):

$$F = 2\hbar k\gamma \frac{\Omega^2}{\delta^2}\left[2(N_+ - N_-) + \frac{1}{3}(N_- - N_+) + \frac{2}{3}\frac{\delta}{\gamma}\operatorname{Im} v\right], \tag{112}$$

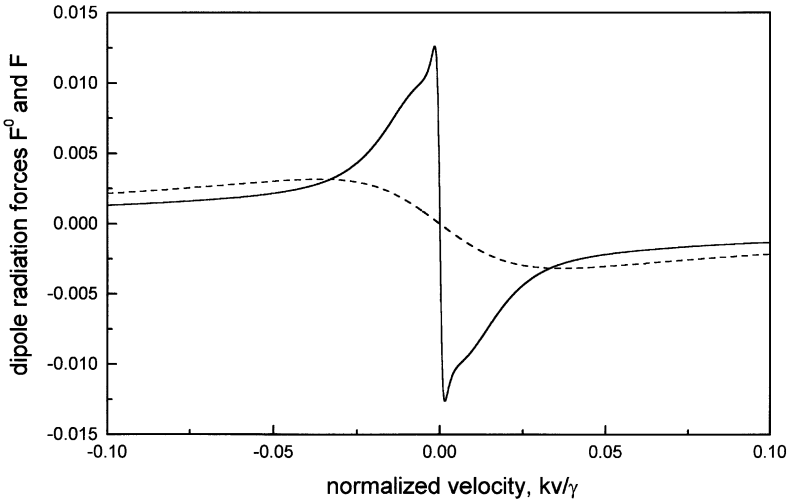

Fig. 15. Spatially averaged radiation force $F^0/\hbar k\gamma$ (solid line) on a (3+5)-level atom in a lin⊥lin configuration and radiation force $F/\hbar k\gamma$ on a (3+5)-level atom (dashed line) for the first $\sigma^+ - \sigma^-$ configuration as functions of velocity $v = v_z$ for the same parameters as in Fig. 14.

where spatially uniform populations $N_\pm = \rho_{g_{\pm 1}g_{\pm 1}}$, $N_0 = \rho_{g_0 g_0}$ and ground-state coherence $\nu = \sigma_{g_{-1}g_1}$ are defined by Eqs. (89) and (91). Note that the Rabi frequency Ω entering Eqs. (111) and (112) is specified by Eq. (107). The above equations show that while the force F for a $\sigma^+ - \sigma^-$ configuration includes a single two-photon structure related to a single spatially uniform ground-state coherence ν, the force F^0 for a lin⊥lin configuration includes the structures related to both the spatially uniform coherence ν^0 and the second harmonics $\nu^{\pm 2}$ of the ground-state coherence. Analytical solutions show specifically that the spatially uniform coherence ν^0 produces the narrow structure and the second harmonics $R_\pm^{\pm 2}$, $\nu^{\pm 2}$ of the ground-state population and coherence produce the super-narrow structures in the force (111). Fig. 16 shows the velocity dependence of a zero harmonic of the dipole radiation force on a (3+5)-level atom in a lin⊥lin configuration in comparison with that of the force for a single $\sigma^+ - \sigma^-$ configuration. As can be seen from Fig. 15 even-order multiphoton processes considerably increase the slope of the force at zero velocity thus producing higher friction coefficient as compared with that for a single $\sigma^+ - \sigma^-$ configuration.

For a motionless or sufficiently slowly moving atom important features of atomic dynamics are related to the spatial variation of the radiation force. For spatially periodic laser field (105) with intensity

$$I = I_0(1 + \cos 2kz \cos 2\omega t)$$

the force at zero velocity, $F(z, 0)$, is also a periodic function of atomic position. The periodic force accordingly produces a periodic potential for a slowly moving atom,

$$U = U(z) = -\int_0^z F(z, 0)\,\mathrm{d}z\ .$$

For a multilevel atom, the spatial periods of the atomic functions are generally different from the period of the field intensity due to the contributions of the multiphoton processes. In particular, for

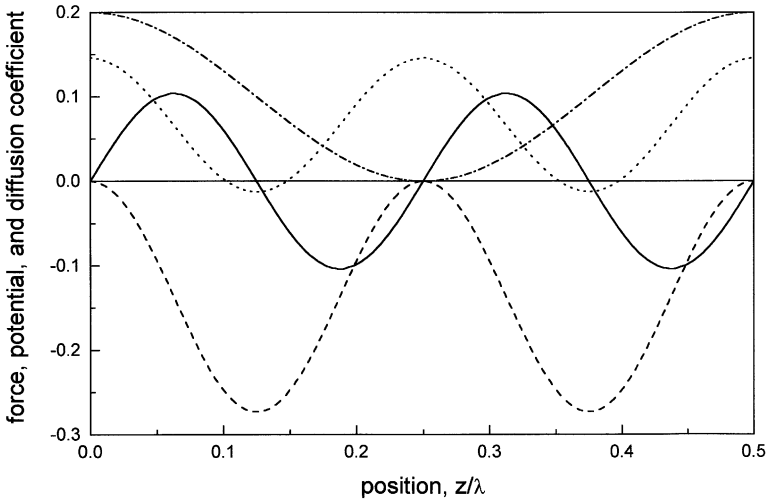

Fig. 16. Dipole radiation force $F/\hbar k\gamma$ (solid line), potential U (dashed line), and diffusion coefficient $0.1D_{zz}/\hbar^2k^2\gamma$ (dotted line) for a motionless (3+5)-level atom in a lin⊥lin field configuration as functions of position for detuning $\delta = -\gamma$ and saturation parameter $G = 1$. All functions are calculated in 4th order REA. Small negative values of the diffusion coefficient are caused by insufficient accuracy of the 4th order REA. For comparison, the dashed-dotted line shows the intensity of the lin⊥lin field, $0.1I/I_0$, as a function of position at time $t = 0$.

a (3+5)-level atom the period of the field intensity is $\Delta z = \pi/k = \lambda/2$ while the period of the force $F(z, 0)$ and potential $U(z)$ is $\lambda/4$ (Fig. 16).

Periodic potentials produced by the radiation forces nowadays find applications for creating periodic lattices of trapped atoms called optical lattices (Jessen and Deutsch, 1996). Some examples of the forces on the multilevel atoms in spatially periodic laser fields and corresponding periodic potentials are discussed in Nienhuis et al. (1991) and Prudnikov et al. (1999).

4. Quantum-kinetic equations

The semiclassical approach considered in Sections 2 and 3 gives a classical description of translational atomic dynamics in laser fields. In some cases the classical approach to atomic dynamics is insufficient since it does not take into account the quantum-mechanical exchange of momentum between the atomic wave packet and the laser and vacuum fields. In order to include into consideration the momentum exchange one has to consider a fully quantum-mechanical description of both the internal and translational atomic state. A proper generalization of the semiclassical density matrix equations to a fully quantum case is given by the quantum-kinetic equations for the atomic density matrix. Similar to semiclassical equations the quantum-kinetic equations include two basic parts, the dynamic and stochastic part. The dynamic part describes pure quantum-mechanical time evolution of the atom in a laser field. The stochastic part describes the influence of a vacuum photon field on atomic dynamics. The dynamic part of the quantum-kinetic equations can be derived straightforwardly from the Schrödinger equation describing the dipole

interaction of the atomic wave packet with the laser field. The stochastic terms entering the quantum-kinetic equations can be derived by applying the Weisskopf–Wigner procedure to a whole system "atomic wave packet+laser field+quantized vacuum field". Below we describe the basic steps leading to the quantum-kinetic equations and discuss the structure of the equations. General equations of this section are specified in Section 5 for some basic interaction schemes.

4.1. Coordinate representation

An atomic wave packet interacting with the laser field can be considered as a quantum-mechanical system that consists of two subsystems. One subsystem includes a set of coordinates $\xi = \xi_1, \ldots, \xi_n$ which describe the internal atomic motion, i.e. the motion of the electrons in the atom and the motion of nucleus. The second subsystem includes the variable of atomic coordinate $\mathbf{r}$ that describes the motion of the atomic wave packet in space. When the interaction of the atom with a vacuum field is neglected the time evolution of the atomic wave packet is described by the wave function $\Psi(\mathbf{r}, \xi, t)$. Atomic wave function $\Psi(\mathbf{r}, \xi, t)$ describes both the internal atomic motion and the translational motion of the atomic wave packet and thus depends on both the internal coordinates ξ and the "external" coordinate $\mathbf{r}$. The Hamiltonian describing the time evolution of the atom in the laser field is generally a function of both the center-of-mass coordinate $\mathbf{r}$ and the internal coordinates ξ, $H = H(\mathbf{r}, \xi)$.

When interaction with a vacuum field is neglected, the atomic density matrix function ρ in the coordinate representation is defined by a product of the wave function $\Psi(\mathbf{r}, \xi, t)$ and the complex conjugate wave function $\Psi^*(\mathbf{r}', \xi', t)$ taken at two different sets of atomic coordinates, $\mathbf{r}, \xi$ and $\mathbf{r}', \xi'$,

$$\rho = \rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t) = \Psi(\mathbf{r}, \xi, t)\Psi^*(\mathbf{r}', \xi', t) \; . \tag{113}$$

Eq. (113) generalizes the definition of the semiclassical density matrix function given by Eq. (5). The atomic density matrix function satisfies the Hermiticity condition

$$\rho^*(\mathbf{r}, \xi; \mathbf{r}', \xi', t) = \rho(\mathbf{r}', \xi'; \mathbf{r}, \xi, t) \tag{114}$$

and the normalization condition following from the normalization condition for the atomic wave function,

$$\int \rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t)_{\substack{\mathbf{r}'=\mathbf{r} \\ \xi'=\xi}} \, \mathrm{d}^3 r \, \mathrm{d}^3 \xi = 1 \; . \tag{115}$$

The equation of motion for the atomic density matrix function $\rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t)$ in the absence of the spontaneous emission follows from the Schrödinger equation. Taking into account the fact that the equation of motion for the atomic density matrix function generally includes the contributions coming from the interaction with a vacuum field here we write out an equation for the function ρ adding an implicit operator term $\Gamma\rho$ describing the effect of the spontaneous relaxation,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t) = H(\mathbf{r}, \xi)\rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t) - H^*(\mathbf{r}', \xi')\rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t) + \mathrm{i}\hbar\Gamma\rho(\mathbf{r}, \xi; \mathbf{r}', \xi', t) \; . \tag{116}$$

In problems related to atomic dynamics in laser fields the Hamiltonian $H(\mathbf{r},\boldsymbol{\xi})$ usually includes three terms, a proper atomic Hamiltonian $H_{\mathrm{a}}(\boldsymbol{\xi})$ that describes the quantized internal atomic states, the kinetic energy operator $K(\mathbf{r}) = -(\hbar^2/2M)\triangle_{\mathbf{r}}$, and the dipole interaction operator $V(\mathbf{r},\boldsymbol{\xi},t) = -\mathbf{d}(\boldsymbol{\xi})\cdot\mathbf{E}(\mathbf{r},t)$,

$$H(\mathbf{r},\boldsymbol{\xi}) = H_{\mathrm{a}}(\boldsymbol{\xi}) + K(\mathbf{r}) + V(\mathbf{r},\boldsymbol{\xi},t)\ . \tag{117}$$

In the interaction representation (2), the decomposition of the total atomic wave function over the stationary time-dependent eigenfunctions (2) is given by

$$\Psi(\mathbf{r},\boldsymbol{\xi},t) = \sum a_k(\mathbf{r})\Psi_k(\boldsymbol{\xi},t)\ , \tag{118}$$

and the atomic density matrix function in the form of decomposition is as follows:

$$\rho(\mathbf{r},\boldsymbol{\xi};\mathbf{r}',\boldsymbol{\xi}',t) = \sum_{m,n}\rho_{mn}(\mathbf{r},\mathbf{r}')\Psi_m(\boldsymbol{\xi},t)\Psi_n^*(\boldsymbol{\xi}',t)\ , \tag{119}$$

where the density matrix elements are

$$\rho_{mn}(\mathbf{r},\mathbf{r}') = a_m(\mathbf{r})a_n^*(\mathbf{r}')\ . \tag{120}$$

Note that the time argument of the density matrix elements is omitted for shortness.

The equations of motion for the atomic density matrix elements are as follows:

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial t}\rho_{kl}(\mathbf{r},\mathbf{r}') = &-\frac{\hbar^2}{2M}(\triangle_{\mathbf{r}} - \triangle_{\mathbf{r}'})\rho_{kl}(\mathbf{r},\mathbf{r}') + \sum_m V_{km}(\mathbf{r},t)\rho_{ml}(\mathbf{r},\mathbf{r}')\\ &- \sum_n \rho_{kn}(\mathbf{r},\mathbf{r}')V_{nl}(\mathbf{r}',t) + \mathrm{i}\hbar\langle k|\Gamma\rho(\mathbf{r},\mathbf{r}')|l\rangle\ ,\end{aligned} \tag{121}$$

where the last implicit term describes as before the contribution of the spontaneous relaxation. Note also that the Laplace operators $\triangle_{\mathbf{r}} = \boldsymbol{\nabla}_{\mathbf{r}}^2$ and $\triangle_{\mathbf{r}'} = \boldsymbol{\nabla}_{\mathbf{r}'}^2$ act accordingly on coordinates $\mathbf{r}$ and $\mathbf{r}'$. In the above equations the matrix elements of the interaction operator $V(\mathbf{r},\boldsymbol{\xi})$ are defined with respect to the time-dependent eigenfunctions,

$$V_{kl}(\mathbf{r},t) = \int \Psi_k^*(\boldsymbol{\xi},t)V(\mathbf{r},\boldsymbol{\xi})\Psi_l(\boldsymbol{\xi},t)\,\mathrm{d}^3\xi = V_{kl}(\mathbf{r})\mathrm{e}^{\mathrm{i}\omega_{kl}t}\ , \tag{122}$$

$$V_{kl}(\mathbf{r}) = \langle k|V(\mathbf{r})|l\rangle = \int \psi_k^*(\boldsymbol{\xi})V(\mathbf{r},\boldsymbol{\xi})\psi_l(\boldsymbol{\xi})\,\mathrm{d}^3\xi\ , \tag{123}$$

where quantities $\omega_{kl} = (E_k - E_l)/\hbar$ may have any sign.

When the atomic wave function is decomposed over the time-independent eigenfunctions $\psi_k(\boldsymbol{\xi})$,

$$\Psi(\mathbf{r},\boldsymbol{\xi},t) = \sum \tilde{a}_k(\mathbf{r})\psi_k(\boldsymbol{\xi})\ , \tag{124}$$

with $\tilde{a}_k(\mathbf{r}) = a_k(\mathbf{r})\exp(-\mathrm{i}E_kt/\hbar)$; the decomposition of the density matrix function has the form of Eq. (13),

$$\rho(\mathbf{r},\boldsymbol{\xi};\mathbf{r}',\boldsymbol{\xi}',t) = \sum_{m,n}\tilde{\rho}_{mn}(\mathbf{r},\mathbf{r}')\psi_m(\boldsymbol{\xi})\psi_n^*(\boldsymbol{\xi}')\ . \tag{125}$$

The density matrix elements $\tilde{\rho}_{mn}$ are

$$\tilde{\rho}_{mn}(\mathbf{r},\mathbf{r}') = \tilde{a}_m(\mathbf{r})\tilde{a}_n^*(\mathbf{r}') \tag{126}$$

and related to density matrix elements defined before as

$$\tilde{\rho}_{mn}(\mathbf{r},\mathbf{r}') = \rho_{mn}(\mathbf{r},\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}\omega_{mn}t}\ . \tag{127}$$

The equations of motion for the density matrix elements $\tilde{\rho}_{mn}$ explicitly include the energy terms,

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial t}\tilde{\rho}_{kl}(\mathbf{r},\mathbf{r}') = &-\frac{\hbar^2}{2M}(\triangle_{\mathbf{r}} - \triangle_{\mathbf{r}'})\tilde{\rho}_{kl}(\mathbf{r},\mathbf{r}') + (E_k(\mathbf{r}) - E_l(\mathbf{r}'))\tilde{\rho}_{kl}(\mathbf{r},\mathbf{r}')\\ &+\sum_m V_{km}(\mathbf{r})\tilde{\rho}_{ml}(\mathbf{r},\mathbf{r}') - \sum_n \tilde{\rho}_{kn}(\mathbf{r},\mathbf{r}')V_{nl}(\mathbf{r}') + \mathrm{i}\hbar\langle k|\Gamma\tilde{\rho}(\mathbf{r},\mathbf{r}')|l\rangle\ ,\end{aligned} \tag{128}$$

where the matrix elements of the interaction operator are defined with respect to the time-independent eigenfunctions according to Eq. (123). The use of the density matrix elements $\tilde{\rho}_{mn}$ may have an advantage in cases when the atomic energies depend on the position of the atom in an external field, $E_n = E_n(\mathbf{r})$, i.e. when a proper atomic Hamiltonian parametrically depends on the atomic center-of-mass coordinate, $H_{\mathrm{a}}(\mathbf{r},\boldsymbol{\xi})$.

4.2. Wigner representation

Along with the coordinate representation the density matrix may also be considered in the momentum representation and in the mixed coordinate-momentum or the Wigner representation (Wigner, 1932; Tatarskii, 1983; Hillery et al., 1984; Balazs and Jennings, 1984). In applications related with the control of the atomic motion by laser fields the most useful is the Wigner representation.

The Wigner representation considers the atomic density matrix as a function of variables of atomic position $\mathbf{r}$ and momentum $\mathbf{p}$. The Wigner representation can be introduced through the coordinate representation with the use of the Fourier transform as

$$\rho(\mathbf{r},\mathbf{p}) = (2\pi\hbar)^{-3/2}\int \rho\left(\mathbf{r}+\frac{1}{2}\mathbf{s},\mathbf{r}-\frac{1}{2}\mathbf{s}\right)\mathrm{e}^{-\mathrm{i}ps/\hbar}\,\mathrm{d}^3s\ . \tag{129}$$

The inverse Fourier transformation is

$$\rho\left(\mathbf{r}+\frac{1}{2}\mathbf{s},\mathbf{r}-\frac{1}{2}\mathbf{s}\right) = (2\pi\hbar)^{-3/2}\int \rho(\mathbf{r},\mathbf{p})\mathrm{e}^{\mathrm{i}ps/\hbar}\,\mathrm{d}^3p\ . \tag{130}$$

The normalization condition for the Wigner density matrix $\rho(\mathbf{r},\mathbf{p})$ follows from the normalization condition (115)

$$\int w(\mathbf{r},\mathbf{p})\,\mathrm{d}^3r\,\mathrm{d}^3p = 1\ , \tag{131}$$

where the quasiprobability distribution function $w(\mathbf{r},\mathbf{p}) = \sum \rho_{kk}(\mathbf{r},\mathbf{p})$ is called the Wigner function.

In order to transform Eqs. (121) and (128) to the Wigner representation one has to represent the interaction terms (122) and (123) in the form of the Fourier expansion,

$$V_{kl}(\mathbf{r},t) = (2\pi)^{-3/2}\int V_{kl}(\mathbf{q},t)\mathrm{e}^{\mathrm{i}qr}\,\mathrm{d}^3q\ , \tag{132}$$

$$V_{kl}(\mathbf{r}) = (2\pi)^{-3/2} \int V_{kl}(\mathbf{q}) \mathrm{e}^{\mathrm{i}qr}\, \mathrm{d}^3 q \ . \tag{133}$$

The substitution of the above Fourier expansions into Eqs. (121) and (128) then gives the equations for the Wigner density matrix elements $\rho_{kl}(\mathbf{r},\mathbf{p})$ and $\tilde{\rho}_{kl}(\mathbf{r},\mathbf{p})$.

4.2.1. Dynamic terms

When the atomic density matrix is defined with respect to the time-dependent atomic eigenfunctions the transformation of Eqs. (121) with the use of Eqs. (129) and (130) gives the equations of motion for the atomic density matrix elements in the Wigner representation as,

$$\begin{aligned} \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho_{kl}(\mathbf{r},\mathbf{p}) = (2\pi)^{-3/2} \sum_m \int V_{km}(\mathbf{q},t)\rho_{ml}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{q}\right)\mathrm{e}^{\mathrm{i}qr}\,\mathrm{d}^3 q \\ -(2\pi)^{-3/2}\sum_n \int \rho_{kn}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{q}\right)V_{nl}(\mathbf{q},t)\mathrm{e}^{\mathrm{i}qr}\,\mathrm{d}^3 q + \mathrm{i}\hbar\langle k|\Gamma\rho(\mathbf{r},\mathbf{p})|l\rangle \ , \end{aligned} \tag{134}$$

where the total time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}, \quad \mathbf{v} = \mathbf{p}/M \ , \tag{135}$$

formally coincides with a classical total time derivative defined by Eq. (15). The dipole interaction terms in the above equations are defined by the Fourier expansions,

$$V_{kl}(\mathbf{q},t) = (2\pi)^{-3/2} \int V_{kl}(\mathbf{r},t)\,\mathrm{e}^{-\mathrm{i}qr}\,\mathrm{d}^3 r \ . \tag{136}$$

When the multilevel atom interacts with the laser field (18) composed of the monochromatic plane waves the equations of motion for the Wigner atomic density matrix elements in the RWA can be written as,

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho_{kl}(\mathbf{r},\mathbf{p}) = &-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^a)\rho_{ml}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{km})t} \\ &+\sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^a)\rho_{kn}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}(\omega_a-\omega_{nl})t} \\ &-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^{a*})\rho_{ml}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{mk})t} \\ &+\sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^{a*})\rho_{kn}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}(\omega_a-\omega_{ln})t} + \mathrm{i}\hbar\langle k|\Gamma\rho(\mathbf{r},\mathbf{p})|l\rangle \ , \end{aligned} \tag{137}$$

where all four sums are assumed to include the terms with positive atomic transition frequencies only, $\omega_{pq}=(E_p-E_q)/\hbar > 0$. The above RWA equations generalize the semiclassical RWA equations (19) and (61).

When the atomic density matrix is defined with respect to the time-independent atomic eigenfunctions the transformation of Eqs. (128) to the Wigner representation gives the equations:

$$\begin{aligned}
\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_{kl}(\mathbf{r},\mathbf{p}) &= (2\pi)^{-3}\int\left[E_k\left(\mathbf{r}+\frac{1}{2}\mathbf{s}\right)-E_l\left(\mathbf{r}-\frac{1}{2}\mathbf{s}\right)\right]\tilde{\rho}_{kl}(\mathbf{r},\mathbf{p}+\hbar\mathbf{q})\mathrm{e}^{\mathrm{i}qs}\,\mathrm{d}^3s\,\mathrm{d}^3q \\
&+(2\pi)^{-3/2}\sum_m\int V_{km}(\mathbf{q})\tilde{\rho}_{ml}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{q}\right)\mathrm{e}^{\mathrm{i}qr}\,\mathrm{d}^3q \\
&-(2\pi)^{-3/2}\sum_n\int\tilde{\rho}_{kn}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{q}\right)V_{nl}(\mathbf{q})\mathrm{e}^{\mathrm{i}qr}\,\mathrm{d}^3q+\mathrm{i}\hbar\langle k|\Gamma\tilde{\rho}(\mathbf{r},\mathbf{p})|l\rangle\ , \qquad (138)
\end{aligned}$$

where

$$V_{kl}(\mathbf{q})=(2\pi)^{-3/2}\int V_{kl}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}qr}\,\mathrm{d}^3r\ . \qquad (139)$$

The above equations differ from Eqs. (134) by the additional energy terms. The internal atomic energies E_k may depend on atomic position $\mathbf{r}$ if a proper atomic Hamiltonian includes the center-of-mass coordinate dependence, $H_\mathrm{a}=H_\mathrm{a}(\mathbf{r},\boldsymbol{\xi})$.

For a practically important case when the multilevel atom interacts with field (18) composed of the monochromatic plane waves Eqs. (138) can be written in the RWA as,

$$\begin{aligned}
\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_{kl}(\mathbf{r},\mathbf{p}) &= (2\pi)^{-3}\int\left[E_k\left(\mathbf{r}+\frac{1}{2}\mathbf{s}\right)-E_l\left(\mathbf{r}-\frac{1}{2}\mathbf{s}\right)\right]\tilde{\rho}_{kl}\,(\mathbf{r},\mathbf{p}+\hbar\mathbf{q})\,\mathrm{e}^{\mathrm{i}qs}\,\mathrm{d}^3s\,\mathrm{d}^3q \\
&-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^a)\tilde{\rho}_{ml}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}\omega_at} \\
&+\sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^a)\tilde{\rho}_{kn}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{\mathrm{i}k_a\mathbf{r}-\mathrm{i}\omega_at} \\
&-\sum_{a,m}(\mathbf{d}_{km}\cdot\mathbf{E}^{a*})\tilde{\rho}_{ml}\left(\mathbf{r},\mathbf{p}+\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}\omega_at} \\
&+\sum_{a,n}(\mathbf{d}_{nl}\cdot\mathbf{E}^{a*})\tilde{\rho}_{kn}\left(\mathbf{r},\mathbf{p}-\frac{1}{2}\hbar\mathbf{k}_a\right)\mathrm{e}^{-\mathrm{i}k_a\mathbf{r}+\mathrm{i}\omega_at}+\mathrm{i}\hbar\langle k|\Gamma\tilde{\rho}(\mathbf{r},\mathbf{p})|l\rangle\ , \qquad (140)
\end{aligned}$$

where the first sum includes the terms with the atomic transition frequencies $\omega_{km}=(E_k-E_m)/\hbar>0$, the second sum includes the terms with frequencies $\omega_{nl}=(E_n-E_l)/\hbar>0$, the third sum includes the terms with frequencies $\omega_{mk}=(E_m-E_k)/\hbar>0$, and the fourth sum includes the terms with frequencies $\omega_{ln}=(E_l-E_n)/\hbar>0$. The representation of the atomic density matrix elements in the form of $\tilde{\rho}_{kl}$ is used in Section 7 for the description of atomic dynamics in a magneto-optical trap.

4.2.2. Spontaneous decay terms

For any specific dipole interaction scheme the general form of the atomic density matrix equations (134) or (138) should be specified with the explicit spontaneous relaxation terms. The derivation of

the relaxation terms for the Wigner density matrix can be done according to the Weisskopf–Wigner procedure described in Section 2.3. For a sufficiently general dipole interaction scheme that includes the hyperfine structure ground-state magnetic sublevels $|\alpha_g F_g M_g\rangle$ and the excited-state magnetic sublevels $|\alpha_e F_e M_e\rangle$ shown in Fig. 1 the terms $(\Gamma\rho)_{kl} = \langle k|\Gamma\rho(\mathbf{r},\mathbf{p},t)|l\rangle$ describing the spontaneous relaxation are:

$$\langle \alpha_{e_1} F_{e_1} M_{e_1}|\Gamma\rho|\alpha_{e_2} F_{e_2} M_{e_2}\rangle = -(\gamma_{\alpha_{e_1} F_{e_1}} + \gamma_{\alpha_{e_2} F_{e_2}})\langle \alpha_{e_1} F_{e_1} M_{e_1}|\rho|\alpha_{e_2} F_{e_2} M_{e_2}\rangle \ ,$$

$$\langle \alpha_e F_e M_e|\Gamma\rho|\alpha_g F_g M_g\rangle = -\gamma_{\alpha_e F_e}\langle \alpha_e F_e M_e|\rho|\alpha_g F_g M_g\rangle \ ,$$

$$\langle \alpha_{g_1} F_{g_1} M_{g_1}|\Gamma\rho|\alpha_{g_2} F_{g_2} M_{g_2}\rangle = \sum_{\alpha_{e_1},\alpha_{e_2},F_{e_1},F_{e_2},M_{e_1},M_{e_2}} \int (F_{g_1} F_{g_2} M_{g_1} M_{g_2}|A(\mathbf{n})|F_{e_1} F_{e_2} M_{e_1} M_{e_2})$$

$$\times \langle \alpha_{e_2} F_{e_2} M_{e_2}|\rho(\mathbf{p} + \mathbf{n}\hbar k)|\alpha_{e_1} F_{e_1} M_{e_1}\rangle \, \mathrm{d}o \ ,$$

$$\langle \alpha_g F_g M'_g|\Gamma\rho|\alpha_g F_g M_g\rangle$$

$$= \sum_{\alpha_e F_e M_e M'_e} \int (F_g M_g M'_g|A(\mathbf{n})|F_e M_e M'_e)\langle \alpha_e F_e M'_e|\rho(\mathbf{p} + \mathbf{n}\hbar k)|\alpha_e F_e M_e\rangle \, \mathrm{d}o \ , \tag{141}$$

where

$$(F_{g_1} F_{g_2} M_{g_1} M_{g_2}|A(\mathbf{n})|F_{e_1} F_{e_2} M_{e_1} M_{e_2})$$

$$= (\gamma_{\alpha_{e_1} F_{e_1},\alpha_{g_1} F_{g_1}} + \gamma_{\alpha_{e_2} F_{e_2},\alpha_{g_2} F_{g_2}}) \sum_{q=0,\pm 1} (F_{g_1} M_{g_1} 1q \,|\, F_{e_1} M_{e_1})(F_{g_2} M_{g_2} 1q \,|\, F_{e_2} M_{e_2})\Phi_q(\mathbf{n})$$

and $(F_g M_g M'_g|A(\mathbf{n})|F_e M_e M'_e) = (F_g F_g M_g M'_g|A(\mathbf{n})|F_e F_e M_e M'_e)$. The functions $\Phi_q(\mathbf{n})$ define the probability of the spontaneous photon emission in the direction of a unit vector $\mathbf{n}$,

$$\Phi_q(\mathbf{n}) = \begin{cases} \frac{3}{8\pi}(1 - n_z^2), & q = 0 \ , \\ \frac{3}{16\pi}(1 + n_z^2), & q = \pm 1 \ , \end{cases} \tag{142}$$

where $n_z = \cos\theta$ is the projection of the unit vector $\mathbf{n}$ on the quantization axis Oz. The integrals in Eqs. (141) are assumed to be taken over the solid angle $\mathrm{d}o = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$. The rest of the terms in Eqs. (141) are the same as in Eqs. (56). When the recoil momentum $\mathbf{n}\hbar k$ is neglected the relaxation terms (141) reduce to the semiclassical expressions (56).

5. Quasiclassical kinetic equations

5.1. Transition to quasiclassical description

In many important problems of atomic dynamics in the laser fields the characteristic relaxation times of the atomic density matrix are much less than the observation time τ_{int}. Owing to fast relaxation of the atomic density matrix to a quasistationary state full quantum-kinetic description

of atomic dynamics can in many cases be reduced to a simpler quasiclassical kinetic description. From the physical point of view the reduction to a quasiclassical description is possible when the relaxation times of the internal atomic states are small compared with that of the translational atomic state. Mathematically, the reduction to the quasiclassical description becomes possible in the cases when the distribution function describing the translational atomic state evolves adiabatically slow as compared with the time evolution of the density matrix elements. When atomic dynamics is described in the Wigner representation the above conditions are satisfied if the relaxation times of the Wigner density matrix elements $\rho_{kl}(\mathbf{r},\mathbf{p},t)$ are short compared with the relaxation time of the Wigner function

$$w = w(\mathbf{r},\mathbf{p},t) = \sum \rho_{kk}(\mathbf{r},\mathbf{p},t) \ , \tag{143}$$

i.e. when the Wigner function $w(\mathbf{r},\mathbf{p},t)$ evolves adiabatically slow as compared with the evolution of the partial Wigner functions $\rho_{kl}(\mathbf{r},\mathbf{p},t)$. The simplification of the initial quantum-kinetic equations to the quasiclassical equations thus becomes possible under the conditions,

$$\tau^1_{\rm rel}, \tau^2_{\rm rel}, \ldots, \ll \tau_{\rm tr} \ , \tag{144}$$

where $\tau^n_{\rm rel}$ are the relaxation times of the atomic density matrix elements and time $\tau_{\rm tr}$ determines the characteristic time scale for the variation of atomic position and velocity. For simple dipole interaction schemes, e.g., for a two-level scheme considered in Section 3.2, the fast relaxation of the internal atomic states is described by the only relaxation time $\tau_{\rm rel} = \tau_{\rm sp} \simeq \gamma^{-1}$ while the variation of the translational motion is defined by an inverse value of the recoil frequency, $\tau_{\rm tr} \simeq \omega_{\rm r}^{-1}$, $\omega_{\rm r} = \hbar k^2/2M$. For more complicated multilevel interaction schemes, e.g., for a (3+5)-level scheme considered in Section 3.3.3, the density matrix elements possess more than one relaxation time.

A standard procedure of the reduction of the equations for the Wigner atomic density matrix elements to the equation for the quasiclassical distribution function $w(\mathbf{r},\mathbf{p},t)$ includes generally the steps which follow the Bogolyubov procedure (Bogolyubov, 1967). In the first step an explicit time and coordinate dependence is excluded from the density matrix equations. Typically this procedure is related with the substitutions for the off-diagonal density matrix elements ρ_{ab}. The substitutions transform the initial equations to new equations for the diagonal elements ρ_{aa} and new off-diagonal elements σ_{ab} where indices a and b may differ from the atomic state indices k and l. Assuming next in accordance with the long observation time that the atom scatters many photons, the momentum width of the atomic density matrix elements can be considered to exceed the photon momentum $\hbar k$. This principal assumption that should be checked later allows one to expand the density matrix elements in the powers of the photon momentum $\hbar k$. Considering next the expanded equations in successively increasing orders in the photon momentum $\hbar k$ one can conclude that the diagonal ρ_{aa} and off-diagonal σ_{ab} density matrix elements are the functionals of the Wigner distribution function $w(\mathbf{r},\mathbf{p},t)$. This means that at the interaction time longer than the "fast" relaxation times, $t = \tau_{\rm int} \gg \tau^1_{\rm rel}, \tau^2_{\rm rel}, \ldots$, the time dependence of the density matrix elements is defined by the time dependence of the quasiclassical distribution function,

$$\rho_{aa}(\mathbf{r},\mathbf{p},t) = \rho_{aa}(\mathbf{r},\mathbf{p};w(\mathbf{r},\mathbf{p},t)) \ ,$$

$$\sigma_{ab}(\mathbf{r},\mathbf{p},t) = \sigma_{ab}(\mathbf{r},\mathbf{p};w(\mathbf{r},\mathbf{p},t)) \ .$$

The general structure of the functional dependence can be found directly from the structure of the expanded equations as a series in the momentum derivatives of the distribution function,

$$\rho_{aa} = \left(R^0_{aa} + \frac{1}{2}\hbar k R^1_{aa} + \cdots \right) w + \frac{1}{2}\hbar k (\mathbf{Q}^1_{aa} + \cdots) \frac{\partial w}{\partial \mathbf{p}} + \cdots ,$$

$$\sigma_{ab} = \left(S^0_{ab} + \frac{1}{2}\hbar k S^1_{ab} + \cdots \right) w + \frac{1}{2}\hbar k (\mathbf{T}^1_{ab} + \cdots) \frac{\partial w}{\partial \mathbf{p}} + \cdots , \tag{145}$$

where $R^0_{aa}, R^1_{aa}, \mathbf{Q}^1_{aa}, \ldots, S^0_{ab}, S^1_{ab}, \mathbf{T}^1_{ab}, \ldots$ are the functions of the atomic momentum $\mathbf{p}$ (or atomic velocity $\mathbf{v} = \mathbf{p}/M$) that have to be determined by the solution procedure. In accordance with the definition of the distribution function (143), the unknown "diagonal" atomic functions satisfy the normalization conditions,

$$\sum R^0_{aa} = 1, \quad \sum R^1_{aa} = 0, \quad \sum \mathbf{Q}^1_{aa} = 0, \ldots . \tag{146}$$

The above procedure evidently generalizes the semiclassical description of atomic dynamics discussed in Section 2. The diagonal atomic functions describe the normalized atomic populations $N_a = R^0_{aa}$ for a classically moving atom, while the off-diagonal functions S^0_{ab} describe the atomic coherences. Rest functions $R^1_{aa}, \mathbf{Q}^1_{aa}, \ldots, S^1_{ab}, \mathbf{T}^1_{ab}, \ldots$ entering Eqs. (145) jointly with Planck's constant $\hbar$ contribute to the shape of the atomic wave packet.

General structure of solution (145) is a key to the derivation of the equation for the leading function $w(\mathbf{r}, \mathbf{p}, t)$. Considering the expanded equations in increasing orders in the photon momentum (in orders in Planck's constant) and taking into account an explicit structure of the solution (145) one can derive from the expanded equations the closed equation for the distribution function $w = w(\mathbf{r}, \mathbf{p}, t)$. In zero order in the photon momentum the distribution function satisfies the phase density conservation equation,

$$\frac{\partial w}{\partial t} + \mathbf{v}\frac{\partial w}{\partial \mathbf{r}} = 0 .$$

This equation is physically evident as in the absence of the photon recoil the atom moves freely. When considered to the first order in the photon momentum the expanded density matrix equations reduce to the Liouville equation for the distribution function w,

$$\frac{\partial w}{\partial t} + \mathbf{v}\frac{\partial w}{\partial \mathbf{r}} = -\frac{\partial}{\partial p_z}(Fw) , \tag{147}$$

where the first kinetic coefficient F defines the dipole radiation force already found by the semiclassical approach (see Eq. (65)). To find the rest of the atomic functions to the first order in the photon momentum one has to take into account the total time derivatives for the functions R^0_{aa}, S^0_{ab} to the first order according to the rule from the functional dependence (145) and the first-order kinetic equation (147),

$$\left(\frac{\partial w}{\partial t} + \mathbf{v}\frac{\partial w}{\partial \mathbf{r}} \right) R^0_{aa} = -R^0_{aa} F \frac{\partial w}{\partial p_z} ,$$

$$\left(\frac{\partial w}{\partial t}+\mathbf{v}\frac{\partial w}{\partial \mathbf{r}}\right)S^0_{ab}=-S^0_{ab}F\frac{\partial w}{\partial p_z}\ .$$

Considering finally the expanded equations to the second order in the photon momentum one can derive a Fokker–Planck type kinetic equation for the distribution function (Minogin, 1980, 1981; Javanainen, 1991; Berg-Sørensen et al., 1992; Stenholm, 1986),

$$\frac{\partial w}{\partial t}+\mathbf{v}\frac{\partial w}{\partial \mathbf{r}}=-\frac{\partial}{\partial \mathbf{p}}(\mathbf{F}w)+\sum\frac{\partial^2}{\partial p_i^2}(D_{ii}w)\ , \tag{148}$$

where the kinetic coefficients $\mathbf{F}$ and D_{ii} define the dipole radiation force and the momentum diffusion tensor ($i=x,y,z$). Considerations of the expanded equations to higher-order terms can be shown to give negligibly small corrections to the second-order Fokker–Planck equation (148).

Eq. (148) is thus a final quasiclassical equation for the atomic distribution function $w(\mathbf{r},\mathbf{p},t)$. Note that in the case of a single atom the quasiclassical distribution function is assumed to be normalized "per a single atom" according to Eq. (131). Eq. (148) can also be applied to an ensemble of noninteracting atoms. In that case the distribution function is to be normalized per a total number of atoms.

5.2. Fokker–Planck equation for two-level atom

To illustrate the procedure of the transition to the quasiclassical description and introduce useful notations we first describe the case of interaction of a two-level atom with a laser beam (68). For a two-level atom the Wigner quantum-kinetic equations (134) written in the time-dependent representation and RWA are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{ee}&=\mathrm{i}\Omega(\mathbf{r})(\rho^{(-)}_{ge}\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)}-\rho^{(-)}_{eg}\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\delta t)})-2\gamma\rho_{ee}\ ,\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{eg}&=\mathrm{i}\Omega(\mathbf{r})(\rho^{(-)}_{gg}-\rho^{(+)}_{ee})\,\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)}-\gamma\rho_{eg}\ ,\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{gg}&=\mathrm{i}\Omega(\mathbf{r})(\rho^{(+)}_{eg}\mathrm{e}^{-\mathrm{i}(\mathbf{kr}-\delta t)}-\rho^{(+)}_{ge}\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\delta t)})+2\gamma\langle\rho^{(q)}_{ee}\rangle\ ,
\end{aligned} \tag{149}$$

where the Rabi frequency Ω and detuning δ are defined as in semiclassical equations (54). The density matrix elements in Eqs. (149) are

$$\rho_{ab}=\langle a|\rho(\mathbf{r},\mathbf{p},t)|b\rangle,\quad \rho^{(\pm)}_{ab}=\left\langle a\left|\rho\left(\mathbf{r},\mathbf{p}\pm\frac{1}{2}\hbar\mathbf{k}\right),t\right|b\right\rangle,\quad \rho^{(\mathbf{n})}_{ab}=\langle a|\rho(\mathbf{r},\mathbf{p}+\mathbf{n}\hbar k,t)|b\rangle\ , \tag{150}$$

where $\mathbf{k}=k\mathbf{e}_z$ and $\mathbf{n}$ is a unit vector that defines the direction of the spontaneous photon emission according to Eqs. (142). Notation $\langle\rho^q_{e_ke_l}\rangle$ is adopted for the density matrix elements averaged over the angular distributions of the spontaneous photon emission,

$$\langle\rho^q_{e_ke_l}\rangle=\int\Phi_q(\mathbf{n})\rho^{(\mathbf{n})}_{e_ke_l}\,\mathrm{d}o\ , \tag{151}$$

where $\mathrm{d}o = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. Note that in a two-level interaction model the laser beam is assumed to be linearly or circularly polarized. In the first case $q = 0$ and in the second case $q = \pm 1$.

After substitutions for the off-diagonal elements $\rho_{eg} = \sigma_{eg}\exp(\mathrm{i}\mathbf{k}r - \mathrm{i}\delta t)$ and application of the quasiclassical transition procedure according to Eqs. (143)–(148) the initial quantum-kinetic equations (149) are reduced to the Fokker–Planck equation (148). The latter equation includes the radiation force defined by Eqs. (73)–(75) and the momentum diffusion tensor (Minogin and Letokhov, 1987),

$$D_{ii}^{q} = \frac{1}{2}\hbar^2 k^2 \gamma \frac{G(\mathbf{r})}{1 + G(\mathbf{r}) + (\delta - \mathbf{k}v)^2/\gamma^2}\chi_{ii}^{q} ,$$

$$\chi_{ii}^{q} = \alpha_{ii}^{q} + \delta_{iz}\left(1 + \frac{G(\mathbf{r})[(\delta - \mathbf{k}v)^2/\gamma^2 - 3]}{[1 + G(\mathbf{r}) + (\delta - \mathbf{k}v)^2/\gamma^2]^2}\right). \tag{152}$$

In the above equations the saturation parameter $G(\mathbf{r})$ is defined by Eq. (76). The coefficients α_{ii}^{q} define the probabilities of the spontaneous photon emission in the direction $i = x, y, z$,

$$\alpha_{ii}^{q} = \int n_i^2 \Phi_q(\mathbf{n})\,\mathrm{d}o . \tag{153}$$

For a laser beam propagating along Oz axis and linearly polarized along Ox axes $\alpha_{xx}^{\pi} = 1/5, \alpha_{yy}^{\pi} = \alpha_{zz}^{\pi} = 2/5$. For a circularly polarized laser beam propagating along Oz axis $\alpha_{xx}^{\sigma} = \alpha_{yy}^{\sigma} = 3/10$, $\alpha_{zz}^{\sigma} = 2/5$. The velocity dependence of the diffusion coefficients for a two-level interaction scheme is shown in Fig. 2 jointly with the velocity dependence of the radiation pressure force and the gradient force.

The diffusion tensor D_{ii}^{q} includes two physically different parts. The first part of the diffusion tensor that is proportional to the angular anisotropy coefficients α_{ii}^{q} originates from the fluctuations in the direction of the spontaneous photon emission. The behavior of this part follows the behavior of the upper-state population $n_{ee} = R_{ee}^{0}$. The second part that exists only for the longitudinal diffusion coefficient D_{zz}^{q} comes from the fluctuations in the number of laser photons scattered by the atom (Cook, 1980; Gordon and Ashkin, 1980; Minogin, 1980). The velocity dependence of the second part exhibits a dip at resonance velocity $v_{\mathrm{res}} = \delta/k$ reflecting the correlations in the process of photon scattering (Fig. 2).

Note that for practical evaluations based on a two-level model the angular anisotropy coefficients can be approximated as $\alpha_{xx}^{q} = \alpha_{yy}^{q} = \alpha_{zz}^{q} = 1/3$. This approximation corresponds to a hypothetical isotropic spontaneous emission.

5.3. Fokker–Planck equations for multilevel atoms in $\sigma^+ - \sigma^-$ field configuration

5.3.1. (1+3)-level atom

In the case of excitation of a (1+3)-level atom by two counter-propagating laser waves defined by Eqs. (82) the dipole interaction is described by a V-type excitation scheme (Fig. 3a). For this scheme the structure of the quantum-kinetic equations (134) written in an RWA can be illustrated by some basic equations as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0 g_0} = \mathrm{i}\Omega(\mathrm{e}^{\mathrm{i}kz + \mathrm{i}\delta t}\rho_{e_{-1}g_0}^{(-)} + \mathrm{e}^{-\mathrm{i}kz + \mathrm{i}\delta t}\rho_{e_1 g_0}^{(+)}) + \mathrm{c.c.} + 2\gamma(\langle\rho_{e_{-1}e_{-1}}^{-1}\rangle + \langle\rho_{e_0 e_0}^{0}\rangle + \langle\rho_{e_1 e_1}^{1}\rangle) ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{e_{-1}e_{-1}} = \mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}kz-\mathrm{i}\delta t}\rho^{(+)}_{g_0e_{-1}} + \text{c.c.} - 2\gamma\rho_{e_{-1}e_{-1}} ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0e_{-1}} = -\mathrm{i}\Omega \mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}(\rho^{(-)}_{g_0g_0} - \rho^{(-)}_{e_{-1}e_{-1}}) + \mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_1e_{-1}} - \gamma\rho_{g_0e_{-1}} , \tag{154}$$

where the Rabi frequency Ω is defined by Eq. (84) and the averaging over the spontaneous angular distribution is defined by Eq. (151). The above equations generalize the semiclassical equations (83). An application of the reduction procedure described in Section 5.1 to set (154) gives the dipole radiation force and the momentum diffusion tensor which are basically defined by the one-photon absorption (emission) processes. The velocity dependences of the longitudinal diffusion coefficient D_{zz} for a (1+3)-level atom is shown jointly with that of the radiation force in Fig. 4. Similar to the force the diffusion coefficient has a relatively simple structure formed basically by the one-photon resonances centered at resonance velocities $v_{\mathrm{res}} = \pm\delta/k$. Two relatively narrow dips also located at resonance velocities come from the fluctuations in the number of the laser photons scattered by the atom. Each of these two dips is similar to the dip in the diffusion coefficient for a two-level atom.

At a low optical saturation when the force is defined by Eq. (86) the momentum diffusion tensor calculated also to a second order in a small Rabi frequency Ω (84) is

$$D_{ii} = \frac{(3 + 11\delta_{iz})}{20}\hbar^2k^2\gamma G(L_- + L_+) . \tag{155}$$

This simple equation useful for estimations does not describe the velocity dips shown in Fig. 4. The dips in the longitudinal diffusion coefficient appear when the equation for D_{ii} is considered up to the fourth power in the Rabi frequency.

5.3.2. (3+3)-level atom

For a (3+3)-level atom excited by the field (82) as shown in Fig. 3b, the force is zero at all velocities (see Section 3.3.2). The population of the upper state $|e_0\rangle$ is generally different from zero. Accordingly, the diffusion tensor is generally also different from zero. For this scheme the quantum-kinetic equations generalize the semiclassical equations (87) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_{-1}} = \mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_0g_{-1}} + \text{c.c.} + \gamma(\langle\rho^0_{e_{-1}e_{-1}}\rangle + \langle\rho^1_{e_0e_0}\rangle) ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \mathrm{i}\Omega(\mathrm{e}^{-\mathrm{i}\delta t}\rho^{(-)}_{g_{-1}e_0} + \mathrm{e}^{\mathrm{i}\delta t}\rho^{(+)}_{e_0g_1})\,\mathrm{e}^{-\mathrm{i}kz} - \gamma\langle\rho^0_{e_{-1}e_1}\rangle ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}e_0} = -\mathrm{i}\Omega(\rho^{(-)}_{g_{-1}g_{-1}} - \rho^{(+)}_{e_0e_0})\,\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t} + \mathrm{i}\Omega \mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{g_{-1}g_1} - \gamma\rho_{g_{-1}e_0} . \tag{156}$$

The velocity dependence of the diffusion coefficients D_{ii} for a (3+3)-level atom is similar to that of the upper-state population $n_{e_0e_0} = R^0_{e_0e_0}$ (see Fig. 5). At zero velocity the upper-state population $n_{e_0e_0}$ and accordingly the diffusion coefficients are equal to zero due to the coherent population trapping effect.

5.3.3. (3+5)-level atom

In the case of interaction of a (3+5)-level atom with the laser field composed of two counter-propagating circularly polarized waves (82) the interaction scheme is shown in Fig. 3c. For this case basic

types of quantum-kinetic equations (134) generalizing the semiclassical equations (89) are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} = \frac{\mathrm{i}\Omega}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}kz}\rho^{(-)}_{e_{-1}g_0} + \mathrm{e}^{-\mathrm{i}kz}\rho^{(+)}_{e_1g_0}\right)\mathrm{e}^{\mathrm{i}\delta t} + \text{c.c.} + \gamma\left(\langle\rho^{-1}_{e_{-1}e_{-1}}\rangle + \frac{4}{3}\langle\rho^{0}_{e_0e_0}\rangle + \langle\rho^{1}_{e_1e_1}\rangle\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \mathrm{i}\Omega(\mathrm{e}^{\mathrm{i}\delta t}\rho^{(-)}_{e_{-2}g_1} - \mathrm{e}^{-\mathrm{i}\delta t}\rho^{(+)}_{g_{-1}e_2})\,\mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{i}\Omega}{\sqrt{6}}(\mathrm{e}^{\mathrm{i}\delta t}\rho^{(+)}_{e_0g_1} - \mathrm{e}^{-\mathrm{i}\delta t}\rho^{(-)}_{g_{-1}e_0})\,\mathrm{e}^{-\mathrm{i}kz}$$
$$+\gamma\left(\sqrt{\frac{2}{3}}\langle\rho^{-1}_{e_{-2}e_0}\rangle + \langle\rho^{0}_{e_{-1}e_1}\rangle + \sqrt{\frac{2}{3}}\langle\rho^{1}_{e_0e_2}\rangle\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_1e_2} = -\mathrm{i}\Omega(\rho^{(-)}_{g_1g_1} - \rho^{(+)}_{e_2e_2})\,\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t} + \frac{\mathrm{i}\Omega}{\sqrt{6}}\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(-)}_{e_0e_2} - \gamma\rho_{g_1e_2}\,, \tag{157}$$

where the Rabi frequency Ω is defined by Eq. (90) and the averaging over the spontaneous angular distribution is defined by Eq. (151).

With substitution of Eqs. (91) and application of the procedure of the transition to a quasiclassical description the initial equations for a (3+5)-level scheme reduce to the Fokker–Planck equation (148). The latter includes the dipole radiation force $\mathbf{F} = F\mathbf{e}_z$ defined by Eq. (92) and the momentum diffusion tensor D_{ii},

$$F = 2\hbar k\Omega\,\mathrm{Im}\left[(S^0_{g_{-1}e_{-2}} - S^0_{g_1e_2}) - \frac{1}{\sqrt{2}}(S^0_{g_0e_1} - S^0_{g_0e_{-1}}) + \frac{1}{\sqrt{6}}(S^0_{g_1e_0} - S^0_{g_{-1}e_0})\right],$$

$$D_{ii} = \hbar^2k^2\gamma\left[\alpha^{\sigma}_{ii}\left(R^0_{e_{-2}e_{-2}} + \frac{1}{2}R^0_{e_{-1}e_{-1}} + \frac{1}{3}R^0_{e_0e_0} + \frac{1}{2}R^0_{e_1e_1} + R^0_{e_2e_2}\right)\right.$$
$$\left.+\alpha^{\pi}_{ii}\left(\frac{1}{2}R^0_{e_{-1}e_{-1}} + \frac{2}{3}R^0_{e_0e_0} + \frac{1}{2}R^0_{e_1e_1}\right)\right]$$
$$+\delta_{iz}\hbar^2k^2\Omega\,\mathrm{Im}\left[(T^1_{g_1e_2} + T^1_{e_{-2}g_{-1}}) + \frac{1}{\sqrt{2}}(T^1_{g_0e_1} + T^1_{e_{-1}g_0}) + \frac{1}{\sqrt{6}}(T^1_{g_{-1}e_0} + T^1_{e_0g_1})\right], \tag{158}$$

where the coefficients α^{σ}_{ii}, α^{π}_{ii} are defined by Eq. (153).

The kinetic coefficients F and D_{ii} that govern the time evolution of the distribution function $w(\mathbf{r},\mathbf{p},t)$ can be explicitly determined by solving the steady-state equations that follow from the expanded equations for the atomic density matrix elements considered separately in the zeroth and the first order in the photon momentum $\hbar k$. The steady-state equations of the zeroth order in $\hbar k$ that determine the functions R^0_{aa} and S^0_{ab} naturally coincide with the steady-state equations for the semiclassical density matrix elements ρ_{aa} and σ_{ab} which follow from Eq. (89). The equations for the functions Q^1_{aa} and T^1_{ab} follow from the quantum-kinetic equations partly listed in (157) after the expansion to the first order in $\hbar k$ (Chang et al., 1999b).

Similar to already considered cases, the diffusion tensor D_{ii} includes two physically different parts. The first part of the diffusion tensor that is proportional to the angular anisotropy coefficients α^{σ}_{ii}, α^{π}_{ii} and the upper-state populations $n_\alpha = R^0_{e_\alpha e_\alpha}$ originates from the fluctuations in the direction of the spontaneous photon emission and the second part that is proportional to the optical coherences $T^1_{g_\alpha e_\beta}$ comes from the fluctuations in the number of scattered photons.

In a low-saturation and low-velocity approximation when the force is defined by Eq. (101) the longitudinal diffusion coefficient $D_{zz} = D$ can be written as,

$$D = D_{\rm sp} + D_{\rm ind}\ ,$$

$$D_{\rm sp} = \frac{1}{2}\hbar^2 k^2 \gamma \frac{G}{(1+\delta^2/\gamma^2)}\left(\alpha_\sigma + \left(\alpha_\pi - \frac{1}{4}\alpha_\sigma\right)N_0\right)\ ,$$

$$D_{\rm ind} = \frac{1}{2}\hbar^2 k^2 \gamma \frac{G}{(1+\delta^2/\gamma^2)}\left(\frac{4}{3} - \frac{13}{12}N_0 + \frac{5}{6}(Q_- - Q_+) + \frac{{\rm i}\delta}{6\gamma}(\tau - \tau^*)\right), \tag{159}$$

where $Q_\pm = Q^1_{g_{\pm1}g_{\pm1}}$, $\tau = T^1_{g_{-1}g_1}$. Similar to the dipole radiation force, the diffusion coefficient D includes a narrow two-photon velocity structure located at zero velocity (Fig. 6). This structure can be shown to decrease the value of the diffusion coefficient $D = D_{zz}$ at zero velocity. An analysis of the expanded equations shows that the two-photon processes give different contributions to the two parts of the diffusion coefficient $D = D_{zz}$ represented by Eq. (159). The two-photon velocity structure located at zero velocity increases the first part of the diffusion coefficient, $D_{\rm sp}$, but much more decreases the second part of the diffusion coefficient, $D_{\rm ind}$. The second part of the diffusion coefficient is, however, much bigger than the first one since the contribution of the ground state coherence is directly related to the ground state populations. As a result, the diffusion coefficient D manifests a narrow velocity dip at velocity $v_z = 0$.

5.3.4. (5+3)-level atom

For this interaction scheme shown in Fig. 3d the quantum-kinetic equations generalize the semi-classical equations (103) as

$$\frac{\rm d}{{\rm d}t}\rho_{g_0g_0} = \frac{{\rm i}\Omega}{\sqrt{6}}({\rm e}^{{\rm i}kz}\rho^{(-)}_{e_{-1}g_0} + {\rm e}^{-{\rm i}kz}\rho^{(+)}_{e_1g_0})\,{\rm e}^{{\rm i}\delta t} + {\rm c.c.} + \gamma\left(\frac{1}{5}\langle\rho^\sigma_{e_{-1}e_{-1}}\rangle + \frac{4}{5}\langle\rho^0_{e_0e_0}\rangle + \frac{1}{5}\langle\rho^1_{e_1e_1}\rangle\right),$$

$$\frac{\rm d}{{\rm d}t}\rho_{g_{-1}g_1} = \frac{{\rm i}\Omega}{\sqrt{2}}({\rm e}^{{\rm i}\delta t}\rho^{(+)}_{e_0g_1} - {\rm e}^{-{\rm i}\delta t}\rho^{(-)}_{g_{-1}e_0})\,{\rm e}^{-{\rm i}kz} + \frac{3}{5}\gamma\langle\rho^0_{e_{-1}e_1}\rangle,$$

$$\frac{\rm d}{{\rm d}t}\rho_{e_1g_0} = \frac{{\rm i}\Omega}{\sqrt{6}}(\rho^{(-)}_{g_0g_0} - \rho^{(+)}_{e_1e_1})\,{\rm e}^{{\rm i}kz-{\rm i}\delta t} + {\rm i}\Omega\left(\rho^{(+)}_{g_2g_0} - \frac{1}{\sqrt{6}}\rho^{(-)}_{e_1e_{-1}}\right){\rm e}^{-{\rm i}kz-{\rm i}\delta t} - \gamma\rho_{e_1g_0}\ .$$

The reduction procedure gives in this case the momentum diffusion coefficient D_{zz} shown jointly with the radiation force in Fig. 9. The diffusion coefficient D_{zz} exhibits at zero velocity a narrow dip going down to zero. This behavior of the diffusion coefficient naturally reflects the contribution of the coherent population trapping effect most clearly seen in a similar behavior of the upper-state population $n_{e_0e_0} = R^0_{e_0e_0}$ (Fig. 10).

5.4. Multilevel atoms in lin⊥lin *field configuration*

5.4.1. (3+3)-level atom

For the case of interaction of a (3+3)-level atom with a lin⊥lin field configuration (106) the quantum-kinetic equations (134) written in RWA are similar to Eqs. (156). Some example equations

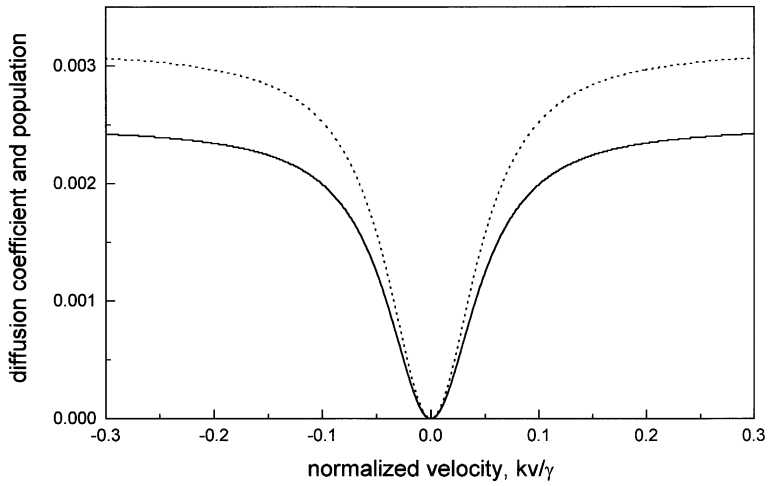

Fig. 17. The velocity dependence of the longitudinal diffusion coefficient $D_{zz}/\hbar^2k^2\gamma$ (dotted line) and upper-state population $n_0=\rho_{e_0e_0}$ (solid line) for a (3+3)-level atom in a lin⊥lin configuration at the detuning $\delta=-20\gamma$ and saturation parameter $G=4$.

for this scheme are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_{-1}}=\Omega(\mathrm{e}^{\mathrm{i}kz}\rho^{(-)}_{e_0g_{-1}}-\mathrm{i}\mathrm{e}^{-\mathrm{i}kz}\rho^{(+)}_{e_0g_{-1}})\,\mathrm{e}^{\mathrm{i}\delta t}+\mathrm{c.c.}+\gamma(\langle\rho^0_{e_{-1}e_{-1}}\rangle+\langle\rho^1_{e_0e_0}\rangle),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1}=\Omega(\mathrm{e}^{\mathrm{i}kz}\rho^{(-)}_{e_0g_1}-\mathrm{i}\mathrm{e}^{-\mathrm{i}kz}\rho^{(+)}_{e_0g_1})\,\mathrm{e}^{\mathrm{i}\delta t}-\Omega(\mathrm{e}^{-\mathrm{i}kz}\rho^{(-)}_{g_{-1}e_0}-\mathrm{i}\mathrm{e}^{\mathrm{i}kz}\rho^{(+)}_{g_{-1}e_0})\,\mathrm{e}^{-\mathrm{i}\delta t}-\gamma\langle\rho^0_{e_{-1}e_1}\rangle\ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}e_0}=\Omega\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}(\rho^{(-)}_{e_0e_0}-\rho^{(+)}_{g_{-1}g_{-1}})-\mathrm{i}\Omega\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}(\rho^{(+)}_{e_0e_0}-\rho^{(-)}_{g_{-1}g_{-1}})$$
$$+\Omega(\mathrm{e}^{\mathrm{i}kz}\rho^{(+)}_{g_{-1}g_1}+\mathrm{i}\mathrm{e}^{-\mathrm{i}kz}\rho^{(-)}_{g_{-1}g_1})\,\mathrm{e}^{\mathrm{i}\delta t}-\gamma\rho_{g_{-1}e_0}\ ,$$

where the Rabi frequency Ω is defined by Eq. (88). For interaction times $\tau_{\mathrm{int}}\gg\gamma^{-1}$ this model reduces to a Λ-type model with zero dipole radiation force. The diffusion tensor is represented by a Fourier series. Spatially averaged value of the diffusion tensor is similar to that for a (3+3)-level atom in a $\sigma^+-\sigma^-$ laser field configuration (Fig. 17).

5.4.2. (3+5)-level atom

In this case the field is represented by Eqs. (105) and (106). The basic types of the Wigner density matrix equations (134) for a (3+5)-level atom in a lin⊥lin laser field configuration are

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0}=\frac{\mathrm{i}\Omega}{\sqrt{2}}(\rho^{(+)}_{e_{-1}g_0}\mathrm{e}^{-\mathrm{i}kz}-\mathrm{i}\rho^{(-)}_{e_{-1}g_0}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t}-\frac{\mathrm{i}\Omega}{\sqrt{2}}(\rho^{(+)}_{e_1g_0}\mathrm{e}^{-\mathrm{i}kz}+\mathrm{i}\rho^{(-)}_{e_1g_0}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t}+\mathrm{c.c}\ .$$
$$+\gamma\left(\langle\rho^{-1}_{e_{-1}e_{-1}}\rangle+\frac{4}{3}\langle\rho^0_{e_0e_0}\rangle+\langle\rho^1_{e_1e_1}\rangle\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \mathrm{i}\Omega(\rho^{(+)}_{e_{-2}g_1}\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\rho^{(-)}_{e_{-2}g_1}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t} - \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho^{(+)}_{e_0g_1}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho^{(-)}_{e_0g_1}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t}$$
$$- \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho^{(+)}_{g_{-1}e_0}\mathrm{e}^{\mathrm{i}kz} + \mathrm{i}\rho^{(-)}_{g_{-1}e_0}\mathrm{e}^{-\mathrm{i}kz})\,\mathrm{e}^{-\mathrm{i}\delta t} + \mathrm{i}\Omega(\rho^{(+)}_{g_{-1}e_2}\mathrm{e}^{\mathrm{i}kz} - \mathrm{i}\rho^{(-)}_{g_{-1}e_2}\mathrm{e}^{-\mathrm{i}kz})\,\mathrm{e}^{-\mathrm{i}\delta t}$$
$$+ \gamma\left(\sqrt{\frac{2}{3}}\langle\rho^{-1}_{e_{-2}e_0}\rangle + \langle\rho^{0}_{e_{-1}e_1}\rangle + \sqrt{\frac{2}{3}}\langle\rho^{1}_{e_0e_2}\rangle\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_1e_2} = \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho^{(+)}_{e_0e_2}\mathrm{e}^{-\mathrm{i}kz} - \mathrm{i}\rho^{(-)}_{e_0e_2}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t} - \mathrm{i}\Omega(\rho^{(+)}_{e_2e_2}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho^{(-)}_{e_2e_2}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t}$$
$$+ \mathrm{i}\Omega(\rho^{(-)}_{g_1g_1}\mathrm{e}^{-\mathrm{i}kz} + \mathrm{i}\rho^{(+)}_{g_1g_1}\mathrm{e}^{\mathrm{i}kz})\,\mathrm{e}^{\mathrm{i}\delta t} - \gamma\rho_{g_1e_2}\ , \tag{160}$$

where the Rabi frequency is defined by Eq. (107).

After expansion of the initial equations partially shown in (160) in the photon recoil and elimination of an explicit time dependence by substitutions (108) for the off-diagonal elements, the expanded functions can be decomposed into infinite series (109). After that the quasiclassical transition procedure gives the Fokker–Planck equation with the dipole radiation force (110) and the momentum diffusion tensor

$$D_{ii} = \hbar^2k^2\gamma\sum_{n=0,\pm2,\ldots} d^n_{ii}\mathrm{e}^{\mathrm{i}nkz}\ ,$$
$$d^n_{ii} = \alpha^{\sigma}_{ii}\left(r^n_{e_{-2}e_{-2}} + \frac{1}{2}r^n_{e_{-1}e_{-1}} + \frac{1}{3}r^n_{e_0e_0} + \frac{1}{2}r^n_{e_1e_1} + r^n_{e_2e_2}\right) + \alpha^{\pi}_{ii}\left(\frac{1}{2}r^n_{e_{-1}e_{-1}} + \frac{2}{3}r^n_{e_0e_0} + \frac{1}{2}r^n_{e_1e_1}\right)$$
$$- \frac{\Omega}{\gamma}\delta_{iz}\left[(T^{n-1}_{e_2g_1} + T^{n-1}_{e_{-2}g_{-1}} - \mathrm{i}T^{n-1}_{g_1e_2} + \mathrm{i}T^{n-1}_{g_{-1}e_{-2}} + T^{n+1}_{g_1e_2} + T^{n+1}_{g_{-1}e_{-2}} + \mathrm{i}T^{n+1}_{e_2g_1} - \mathrm{i}T^{n+1}_{e_{-2}g_{-1}})\right.$$
$$+ \frac{1}{\sqrt{6}}(T^{n-1}_{e_0g_1} + T^{n-1}_{e_0g_{-1}} - \mathrm{i}T^{n-1}_{g_{-1}e_0} + \mathrm{i}T^{n-1}_{g_1e_0} + T^{n+1}_{g_1e_0} + T^{n+1}_{g_{-1}e_0} + \mathrm{i}T^{n+1}_{e_0g_{-1}} - \mathrm{i}T^{n+1}_{e_0g_1})$$
$$\left. + \frac{1}{\sqrt{2}}(T^{n-1}_{e_1g_0} + T^{n-1}_{e_{-1}g_0} - \mathrm{i}T^{n-1}_{g_0e_1} + \mathrm{i}T^{n-1}_{g_0e_{-1}} + T^{n+1}_{g_0e_1} + T^{n+1}_{g_0e_{-1}} + \mathrm{i}T^{n+1}_{e_1g_0} - \mathrm{i}T^{n+1}_{e_{-1}g_0})\right]. \tag{161}$$

The harmonics of the diffusion tensor satisfy the "hermiticity" conditions, $d^{n*}_{ii} = d^{-n}_{ii}$.

The velocity dependence of the momentum diffusion tensor in a low-velocity region is defined by the even-order multiphoton processes considered in Section 3.4.2. Fig. 19 shows the velocity dependence of zero harmonics of the dipole radiation force and longitudinal component of the diffusion tensor for a lin⊥lin configuration in comparison with the radiation force and longitudinal component of the diffusion tensor for a single $\sigma^+ - \sigma^-$ configuration. As can be seen from Fig. 18 even-order multiphoton processes considerably increase the longitudinal component of the diffusion tensor since this quantity is the most sensitive to the fluctuations in the number of scattered photons. While for a $\sigma^+ - \sigma^-$ configuration the longitudinal diffusion coefficient D_{zz} even decreases at zero velocity due to the coherent contribution of a single two-photon process, for a lin⊥lin configuration the longitudinal diffusion coefficient $D^0_{zz} = \hbar^2k^2\gamma d^0_{zz}$ increases at zero velocity reflecting a considerable

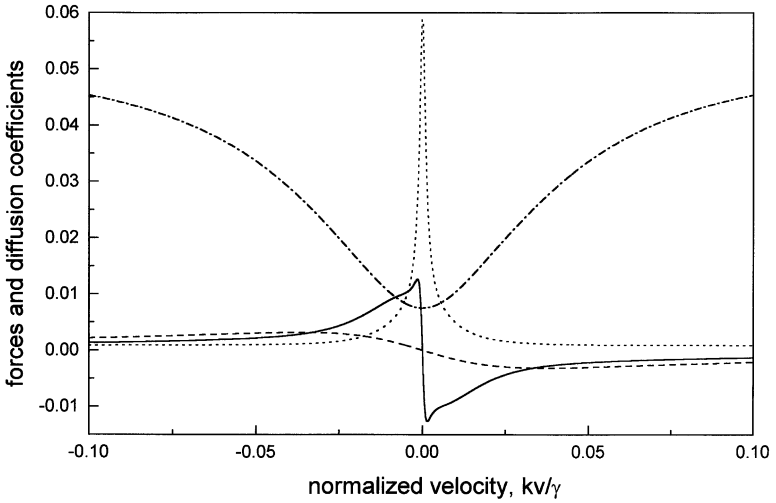

Fig. 18. Spatially averaged radiation force $F^0/\hbar k\gamma$ (solid line) and spatially averaged longitudinal component of the diffusion tensor $0.1D^0_{zz}/\hbar^2k^2\gamma$ (dotted line) for a (3+5)-level atom in a lin⊥lin configuration and radiation force $F/\hbar k\gamma$ (dashed line) and longitudinal component of the diffusion tensor $D_{zz}/\hbar^2k^2\gamma$ (dot-dashed line) for a (3+5)-level atom in the first $\sigma^+ - \sigma^-$ configuration as functions of velocity $v = v_z$ for saturation parameter $G = 2$ and detuning $\delta = -20\gamma$.

increase in the number of scattering channels. Note that in Fig. 18 the value of the diffusion coefficient for a lin⊥lin configuration is plotted in 10 times reduced scale.

Note finally that the coefficients of the Fokker–Planck equation for the multilevel schemes with half integer angular momenta $F_g = F_e = 1/2$ can be found in Prudnikov et al. (1999).

6. Laser cooling of atoms

One of the most important applications of the quasiclassical theory of atomic motion in laser fields is the cooling of atoms by near resonant laser fields (Hänsch and Schawlow, 1975; Wineland and Itano, 1979; Minogin and Letokhov, 1987; Nienhuis et al., 1991; Adams and Riis, 1997; Metcalf and van der Straten, 1999).

Explicit expressions for the coefficients of the Fokker–Planck equation for the dipole interaction schemes which include counter-propagating laser waves can be directly applied for estimating the temperatures achievable in the schemes of laser cooling of atoms. At a negative detuning when the dipole radiation force reduces to a friction force, the Fokker–Planck equation has a steady-state solution describing the stationary velocity distribution of laser-cooled atoms. At low velocities the friction force is a linear function of velocity and the steady-state solution of the Fokker–Planck equation is a Maxwellian distribution defined by the friction produced by the cooling force and the diffusion.

In a simplest one-dimensional case the dipole radiation force playing the role of a cooling force can be represented at low velocities as a friction force

$$F = -M\beta v\ , \tag{162}$$

where β is the friction coefficient. The steady-state velocity distribution for laser-cooled atoms is accordingly defined by the solution of the Fokker–Planck equation as

$$w(v) = \frac{1}{\sqrt{\pi}u} \exp\left(-\frac{v^2}{u^2}\right), \tag{163}$$

where the velocity halfwidth u is related to an effective temperature,

$$u = \sqrt{\frac{2k_B T}{M}}, \tag{164}$$

and k_B is the Boltzmann constant. The effective temperature T is defined by the Einstein relation,

$$T = \frac{D(0)}{M\beta k_B}, \tag{165}$$

where $D(0)$ is the diffusion coefficient at zero velocity.

The value of atomic temperature (165) crucially depends on the types of optical processes which contribute to the friction and diffusion coefficients. Of two coefficients, β and $D(0)$, most important is the behavior of the friction coefficient which is most sensitive to the optical processes. In cases of the dipole interaction schemes not possessing the ground-state coherence as it is the case for a two-level atom or a (1+3)-level atom, the friction force originates from the one-photon resonances which are relatively broad on the velocity scale. The one-photon friction force produces accordingly a relatively low friction and a relatively high value of the temperature known as the Doppler temperature. In the cases of the multilevel dipole interaction schemes possessing the ground-state coherence the two-photon and generally higher-order multiphoton processes considerably increase the friction coefficient. As a result the value of temperature for a multilevel scheme can be below the typical values of the Doppler temperatures. In the latter case the corresponding low temperatures are referred to as the sub-Doppler temperatures (Dalibard and Cohen-Tannoudji, 1989).

6.1. Doppler cooling

The quantitative estimations of the Doppler temperatures can be directly derived from the Fokker–Planck equations for a two-level in the field of two counter-propagating laser waves or a (1+3)-level atom in a $\sigma^+ - \sigma^-$ field configuration. At low saturation the dipole radiation force can be written for both schemes as (see Eqs. (74) and (86)),

$$F = \hbar k \gamma G \left(\frac{1}{1 + (\delta - kv)^2/\gamma^2} - \frac{1}{1 + (\delta + kv)^2/\gamma^2} \right). \tag{166}$$

At a red detuning, $\delta < 0$, the force (166) is directed opposite to atomic velocity. Accordingly, at low velocity the force (166) reduces to a friction force (162) with the friction coefficient

$$\beta = \omega_r \frac{8G|\delta|/\gamma}{(1 + \delta^2/\gamma^2)^2}, \tag{167}$$

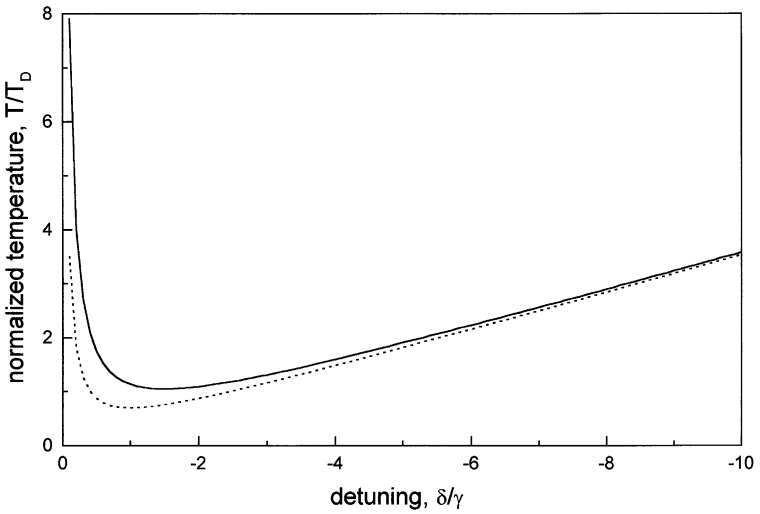

Fig. 19. Atomic temperature for a (1+3)-level atoms in a $\sigma^+ - \sigma^-$ field configuration as a function of detuning at saturation parameter $G = 1$ (solid line). The dotted line shows the temperature (169) valid in a low saturation limit. Temperature is normalized on the Doppler temperature $T_D = \hbar\gamma/k_B$.

where $\omega_r = \hbar k^2/2M$ is the recoil frequency. The longitudinal diffusion coefficient at zero velocity for both schemes is (see Eqs. (152) and (155))

$$D(0) = D_{zz}(0) = \hbar^2 k^2 \gamma \frac{G}{1 + \delta^2/\gamma^2}(1 + \alpha_{zz}) \,. \tag{168}$$

The effective temperature for the Doppler cooling schemes according to Eq. (165) is defined by the value of the detuning and does not depend on the saturation parameter at low effective saturation (Letokhov et al., 1977)

$$T = \frac{1 + \alpha_{zz}}{4} \frac{\hbar\gamma}{k_B} \left(\frac{|\delta|}{\gamma} + \frac{\gamma}{|\delta|} \right). \tag{169}$$

At an optimal detuning $\delta = -\gamma$ the temperature found in a low saturation limit has a minimal value

$$T_{min} = \frac{1 + \alpha_{zz}}{2} T_D, \quad T_D = \frac{\hbar\gamma}{k_B} \,, \tag{170}$$

where the temperature T_D is called the Doppler temperature. Typical dependence of the temperature on the detuning for a (1+3)-level interaction scheme is shown in Fig. 19.

6.2. Sub-Doppler cooling

6.2.1. $\sigma^+ - \sigma^-$ laser field configuration

An estimation of the sub-Doppler temperature can be done on a simplest example of a multilevel atom which possesses the ground-state sublevels—a (3+5)-level atom. In the case of the one-dimensional laser field represented by a $\sigma^+ - \sigma^-$ configuration (82) the dipole radiation force on a (3+5)-level atom at low saturation and low velocities is defined by Eq. (101). At large negative

detunings, $|\delta| \gg \Omega, \gamma$, the friction coefficient is defined by the contributions due to the two-photon processes as

$$\beta = \omega_{\mathrm{r}} \frac{120}{17} \frac{\gamma}{|\delta|} \ . \tag{171}$$

In the same case of large detuning the diffusion coefficient (159) estimated at zero velocity is

$$D(0) \simeq \frac{46}{17} \hbar^2 k^2 \gamma \frac{\Omega^2}{\delta^2} \ . \tag{172}$$

High friction coefficient due to the dipole radiation force and a relatively small value of the diffusion coefficient due to the velocity deep in the momentum diffusion coefficient are finally responsible in the above case for the sub-Doppler laser cooling down to a temperature (165) (Dalibard and Cohen-Tannoudji, 1989; Ungar et al., 1989; Chang et al., 1999a, 2001),

$$T = \frac{23}{30} \frac{\hbar \Omega^2}{k_{\mathrm{B}} |\delta|} \ . \tag{173}$$

The above equation for the temperature is valid under the basic assumption of the kinetic theory that the momentum width of the atomic density matrix elements exceeds the photon momentum $\hbar k$. This principal assumption is always satisfied when the temperature (173) exceeds the recoil temperature, $T > T_{\mathrm{R}} = \hbar \omega_{\mathrm{r}} / k_{\mathrm{B}}$.

More complicated multilevel schemes shown in Fig. 20 exhibit similar sub-Doppler temperatures. For any of these schemes the dipole radiation force includes a narrow structure located at zero velocity. The slope of the dipole radiation force near zero velocity which defines the friction coefficient β increases when the number of levels increases (Fig. 21). The diffusion coefficients $D = D_{zz}$ as functions of velocity show narrow dips located at zero velocity (Fig. 22). Starting from a (5+7)-level atom these dips exhibit additional narrow structures most clearly seen for (7+9)- and (9+11)-level atoms (Fig. 23).

The above behavior of the forces and the diffusion coefficients has a natural explanation in terms of atomic coherences and populations. In cases of more complicated atomic schemes higher even-order multiphoton processes give additional contributions to the ground-state coherences and populations. In the case of a (7+9)-level scheme there are three basic multiphoton processes: 2-photon process contributing to the ground-state coherence $R^0_{g_{-1}g_1}$, 4-photon process contributing to the ground-state coherence $R^0_{g_{-2}g_2}$, and 6-photon process contributing to the ground-state coherence $R^0_{g_{-3}g_3}$. The lowest order 2-photon process gives the most broad velocity structure, the next 4-photon process a medium width structure, and the highest 6-photon process the narrowest structure. This narrowest structure is accordingly responsible for the slope of the force at zero velocity, i.e. for the friction coefficient for a (7+9)-level atom (Fig. 21). In a similar way the above narrowest structure is responsible for the fine structure of the velocity dip in the diffusion coefficient for a (7+9)-level atom (Fig. 22). The same general explanation can be provided for other atomic schemes of Fig. 20.

Figs. 23 and 24 show the dependence of the atomic temperature on the value of a negative detuning and the saturation parameter for the above four atomic schemes. For any given atomic scheme the value of the temperature decreases approximately inverse proportionally to the detuning and proportionally to the saturation parameter. For different schemes the temperature decreases approximately inverse proportionally to the number of levels. All the above features of the temperature behavior reflect the contributions of the multiphoton processes responsible for the friction coefficient

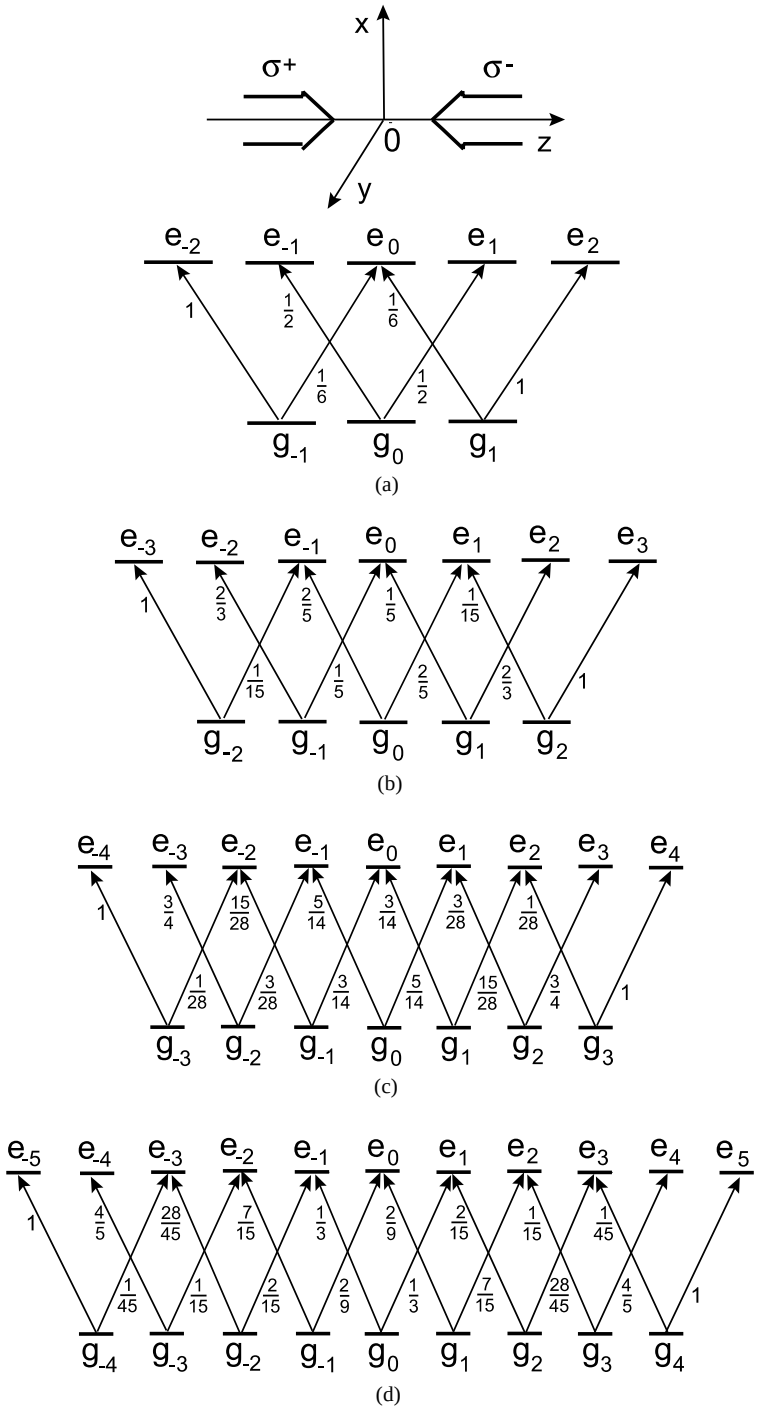

Fig. 20. Schemes of a (3+5)-, (5+7)-, (7+9)- and (9+11)-level atom excited by counter-propagating circularly polarized laser waves composing a $\sigma^+ - \sigma^-$ field configuration. Arrows show the σ^+ ($g_M \to e_{M+1}$) and σ^- ($g_M \to e_{M-1}$) excitation transitions. Numbers show the relative strengths of the dipole $\sigma^\pm$ transitions.

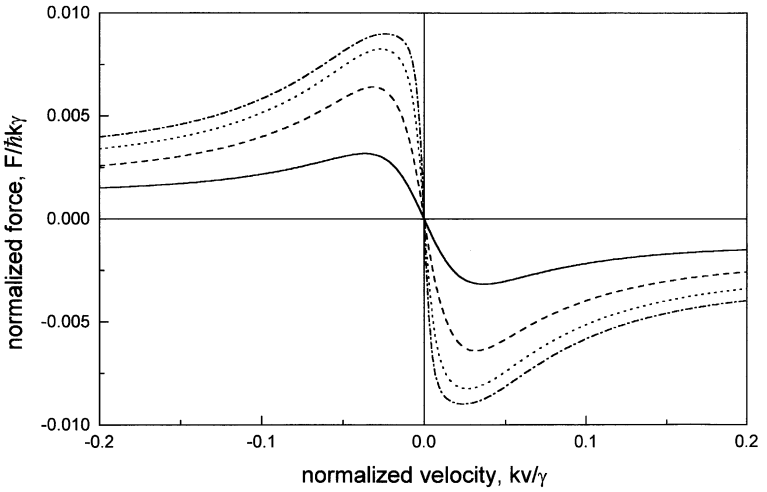

Fig. 21. Dipole radiation force as a function of atomic velocity $v = v_z$ for a (3+5)-level atom (solid line), (5+7)-level atom (dashed line), (7+9)-level atom (dotted line), and (9+11)-level atom (dash-dotted line) excited by a $\sigma^+ - \sigma^-$ field configuration at saturation parameter $G = 2\Omega^2/\gamma^2 = 4$ and detuning $\delta = -20\gamma$.

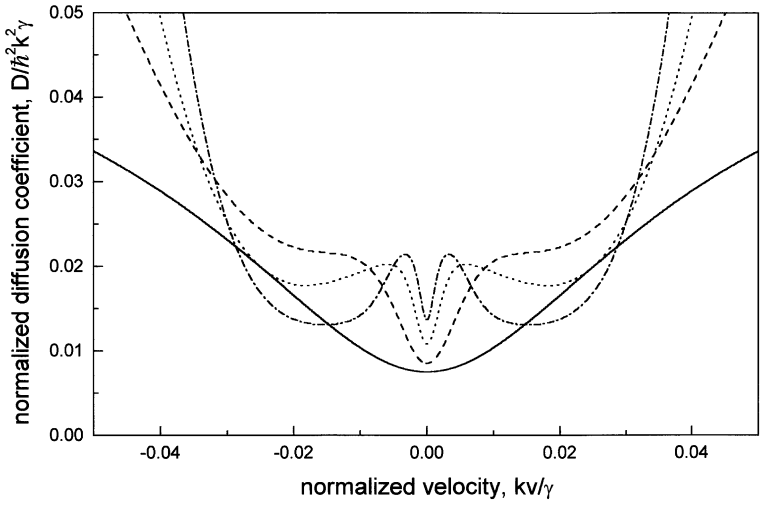

Fig. 22. Detailed structure of the diffusion coefficient $D = D_{zz}$ at small velocities for the same parameters as in Fig. 21.

and the diffusion coefficient at zero velocity. It is to be stressed that the comparisons of different atomic schemes are made above in terms of universally defined dimensionless saturation parameter $G = 2\Omega^2/\gamma^2$. For any practical purposes the saturation parameter can equivalently be represented as $G = I/I_S$, where $I = (c/8\pi)E_0^2$ is the intensity of a single travelling wave with polarization σ^+ or σ^- and the saturation intensity $I_S = \hbar\gamma\omega_0^3/6\pi c^2$ does not explicitly depend on the quantum numbers of the atomic scheme.

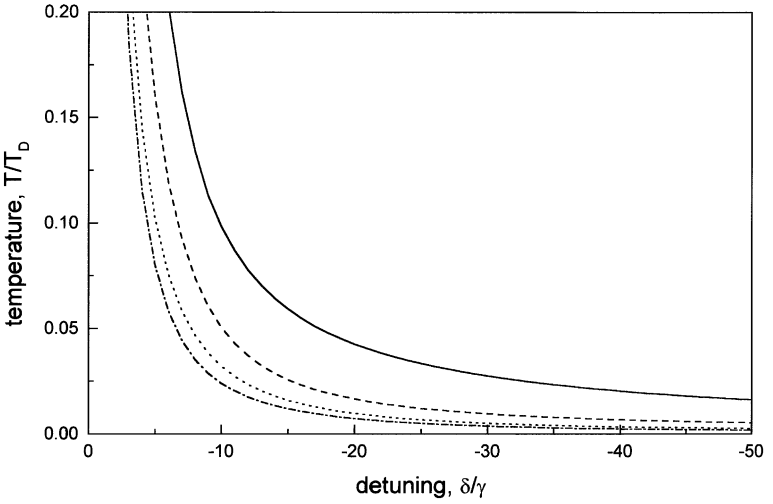

Fig. 23. Atomic temperature as a function of detuning for a (3+5)-level atom (solid line), (5+7)-level atom (dashed line), (7+9)-level atom (dotted line), (9+11)-level atom (dash-dotted line) excited by a $\sigma^+ - \sigma^-$ field configuration at saturation parameter $G = 4$. Temperature is normalized on the Doppler temperature $T_D = \hbar\gamma/k_B$.

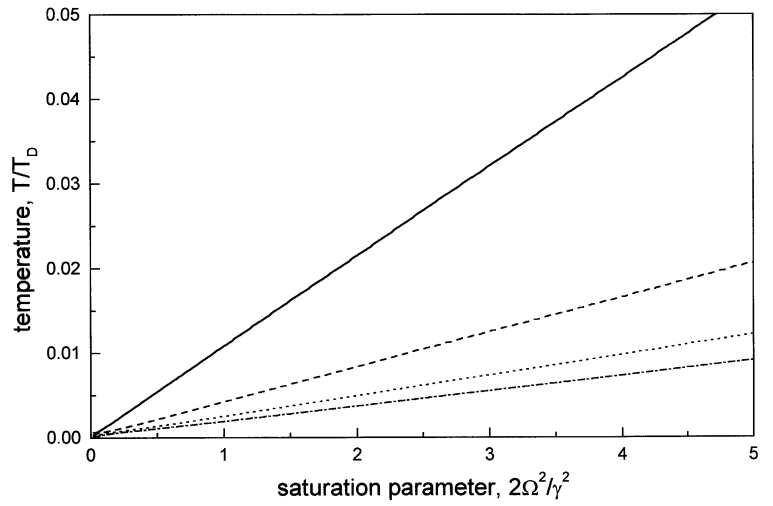

Fig. 24. Atomic temperature as a function of saturation parameter for a (3+5)-level atom (solid line), (5+7)-level atom (dashed line), (7+9)-level atom (dotted line), and (9+11)-level atom (dash-dotted line) in a field of a $\sigma^+ - \sigma^-$ configuration for detuning $\delta = -20\gamma$.

6.2.2. lin⊥lin *laser field configuration*

In the case of a lin⊥lin laser field configuration defined by Eqs. (105) and (106) the value of sub-Doppler temperature is close to that for a $\sigma^+ - \sigma^-$ configuration (Fig. 25). Both considered cases, a $\sigma^+ - \sigma^-$ configuration and a lin⊥lin configuration, thus show very close values of the temperature at a large detuning. This has a natural explanation in terms of the light shift. At a

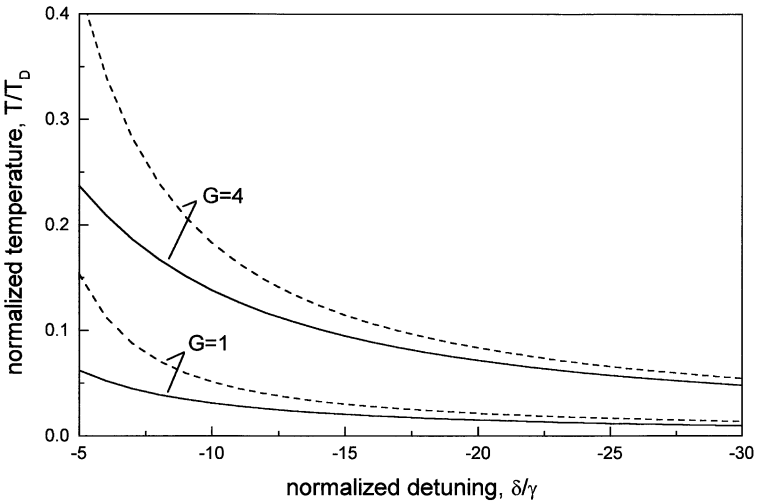

Fig. 25. Atomic temperature as a function of detuning for a lin⊥lin configuration (solid lines) and the first $\sigma^+ - \sigma^-$ configuration (dashed lines) for saturation parameter $G = 1$ and 4. Temperature is normalized on the Doppler temperature $T_D = \hbar\gamma/k_B$.

low effective saturation the entire atomic population is localized on the ground-state sublevels. The internal atomic energy is thus close to zero and the translational atomic temperature is defined by the value of the light shift. For large detunings, $|\delta| \gg \Omega, \gamma$, and small velocities, $k|v| \ll \gamma$, the light shift and accordingly the atomic temperature apart from a numerical factor is

$$E \sim k_B T \sim \hbar \frac{\Omega^2}{|\delta|} .$$

The value of the atomic temperature at a large detuning is thus defined by the same quantity both for a lin⊥lin configuration and a $\sigma^+ - \sigma^-$ configuration.

7. Magneto-optical trap

So far we have discussed the application of the kinetic approach to dynamics of multilevel atoms in pure laser fields. The approach can, however, be used for a quantitative description of atomic motion not only in laser fields but also in the combined fields of the laser fields and other fields. In this section we discuss the application of the kinetic approach to the description of dynamics of multilevel atoms in a magneto-optical trap (MOT) which explores both the laser field and a static magnetic field (Dalibard, 1987; Raab et al., 1987).

Basic features of the MOT capable of operating at sub-Doppler temperatures can be considered on a basic model of a (3+5)-level atom placed in a weak inhomogeneous magnetic field (Fig. 26). This multilevel interaction scheme includes all the basic features common to the real experimental schemes (Drewsen et al., 1994). In the scheme the inhomogeneous magnetic field produces the position-dependent Zeeman shifts. The laser field chosen in a $\sigma^+ - \sigma^-$ configuration (82) is responsible for both one-photon absorption (emission) processes and two-photon optical processes. The

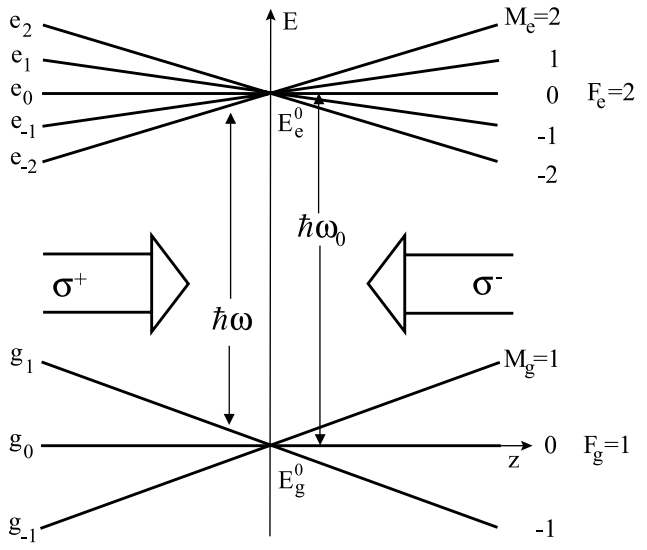

Fig. 26. Zeeman-shifted energy levels for a (3+5)-level atom in a one-dimensional magneto-optical trap. Atoms are placed in a weak inhomogeneous magnetic field and irradiated by a $\sigma^+ - \sigma^-$ field configuration.

frequencies ω of the laser waves are assumed to be red detuned with respect to the frequency of the nonperturbed atomic transition, $\omega < \omega_0$.

We assume below that the atoms are placed in a weak inhomogeneous magnetic field varying along axis Oz,

$$\mathbf{B} = B(z)\mathbf{e}_z, \quad B(z) = az \ , \tag{174}$$

where a is the gradient of the magnetic field in the center of the trap. The laser field is assumed to be near resonance with two hyperfine structure states, the ground state $|\alpha_g, F_g = 1\rangle$ with energy E_g^0 and the excited state $|\alpha_e, F_e = 2\rangle$ with energy E_e^0. Both states are split in the magnetic field over the magnetic sublevels $M_g = 0, \pm 1$ and $M_e = 0, \pm 1, \pm 2$.

7.1. Quantum-kinetic equations

For the considered scheme, the Hamiltonian can be represented as a generalization of the Hamiltonian (117),

$$H = H_{\rm am} - (\hbar^2/2M) \triangle - \mathbf{d} \cdot \mathbf{E} \ ,$$

$$H_{\rm am} = H_{\rm a} - \vec{\mu} \cdot \mathbf{B} \ , \tag{175}$$

where the atomic Hamiltonian $H_{\rm am}$ includes a proper atomic Hamiltonian $H_{\rm a}$ and the term describing the dipole interaction of the atom with inhomogeneous magnetic field (174). The Zeeman shifts of the magnetic substates can be considered for simplicity in a lowest linear approximation in the value

of the magnetic field. For the ground-state and excited-state magnetic sublevels,

$$\begin{aligned}\langle F_g M_g| - \vec{\mu}\cdot\mathbf{B}|F_g M_g\rangle &= \mu_{\mathrm{B}} g_g B(z) M_g \ , \\ \langle F_e M_e| - \vec{\mu}\cdot\mathbf{B}|F_e M_e\rangle &= \mu_{\mathrm{B}} g_e B(z) M_e \ ,\end{aligned} \tag{176}$$

where μ_{B} is the Bohr magneton, g_g and g_e are the Lande g-factors for the ground and excited state and $F_g = 1,\ M_g = 0, \pm 1;\ F_e = 2,\ M_e = 0, \pm 1, \pm 2$.

A natural approach to the description of atomic dynamics in the above scheme is the use of the quantum-kinetic equations (138). In this section we use the notations $\rho_{kl}(\mathbf{r},\mathbf{p})$ for the density matrix elements $\tilde{\rho}_{kl}(\mathbf{r},\mathbf{p})$ entering Eqs. (138), (140) thus omitting the upper "tilde" for simplicity. Before writing the equations we make two simplifications. First, we take into account the fact that the Zeeman-shifted energies vary over a distance much larger than the size of the atomic wave packet and accordingly neglect the integral structure of the energy terms in Eqs. (140). Next, we exclude an explicit time dependence in Eqs. (140) by changing the off-diagonal density matrix elements as

$$\rho_{g_\alpha e_\beta} \to \rho_{g_\alpha e_\beta} \exp \mathrm{i}\omega t \ ,$$

where ω is a frequency of the monochromatic laser field (82). After the above changes the Wigner density matrix equations describing the dipole interaction of a (3+5)-level atom with laser field (82) in the presence of the magnetic field (174) basically differ from Eqs. (157) in terms of the additional energy terms. Some basic types of equations are as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0 g_0} &= \frac{\mathrm{i}\Omega}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}kz}\rho^{(-)}_{e_{-1}g_0} + \mathrm{e}^{-\mathrm{i}kz}\rho^{(+)}_{e_1 g_0})\mathrm{e}^{\mathrm{i}\delta t} + \mathrm{c.c.} + \gamma\left(\langle\rho^{-1}_{e_{-1}e_{-1}}\rangle + \frac{4}{3}\langle\rho^{0}_{e_0 e_0}\rangle + \langle\rho^{1}_{e_1 e_1}\rangle\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} &= \mathrm{i}\Omega(\rho^{(-)}_{e_{-2}g_1} - \rho^{(+)}_{g_{-1}e_2})\mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{i}\Omega}{\sqrt{6}}(\rho^{(+)}_{e_0 g_1} - \rho^{(-)}_{g_{-1}e_0})\mathrm{e}^{-\mathrm{i}kz} \\ &\quad + \gamma\left(\sqrt{\frac{2}{3}}\langle\rho^{-1}_{e_{-2}e_0}\rangle + \langle\rho^{0}_{e_{-1}e_1}\rangle + \sqrt{\frac{2}{3}}\langle\rho^{1}_{e_0 e_2}\rangle\right) + 2\mathrm{i}\omega_g\rho_{g_{-1}g_1} \ , \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_1 e_2} &= -\mathrm{i}\Omega(\rho^{(-)}_{g_1 g_1} - \rho^{(+)}_{e_2 e_2})\mathrm{e}^{-\mathrm{i}kz} + \frac{\mathrm{i}\Omega}{\sqrt{6}}\mathrm{e}^{\mathrm{i}kz}\rho^{(-)}_{e_0 e_2} - (\gamma + \mathrm{i}(\omega_g - 2\omega_e + \delta))\rho_{g_1 e_2} \ .\end{aligned} \tag{177}$$

In the above equations the dipole interaction terms are defined by Eqs. (139) and the spontaneous relaxation terms by Eqs. (141)–(142). The averaging over the spontaneous angular distribution is specified by Eq. (151). The detuning is defined as $\delta = \omega - \omega_0$, where ω_0 is the atomic transition frequency in the absence of the magnetic field. The Zeeman frequency shifts entering the above equations,

$$\omega_g = \mu_{\mathrm{B}} g_g a z/\hbar, \quad \omega_e = \mu_{\mathrm{B}} g_e a z/\hbar \ , \tag{178}$$

determine the position-dependent Larmor frequencies $\omega_{\mathrm{L}g} = |\omega_g|$, $\omega_{\mathrm{L}e} = |\omega_e|$ for the ground and excited states. Depending on signs of the Lande g-factors the Zeeman shifts ω_g and ω_e can be positive or negative. Note that the adopted simplification for the energy terms in Eqs. (177) means that small magnetic dipole forces $\mathbf{f}_k = \partial\langle k|\vec{\mu}\cdot\mathbf{B}|k\rangle/\partial\mathbf{r}$ which do not play any noticeable role in atomic dynamics in the MOT are neglected.

7.2. Dipole radiation force

The procedure of simplification of the quantum-kinetic equations describing the motion of a (3+5)-level atom in the MOT coincides with the procedure of Section 5.3.3 for the case of absence of the magnetic field. Accordingly, the dipole radiation force on a (3+5)-level atom in the MOT is defined by the equation formally identical to Eq. (92) while an explicit equation for the force differs from Eq. (96) by evident change of the detunings. The dipole radiation force on the atom in the MOT is the velocity- and position-dependent quantity. In a physically most interesting case of a low one-photon saturation,

$$s_{p,q,\pm} = \Omega^2/(\gamma^2 + \delta^2_{p,q,\pm}) \ll 1 \ , \tag{179}$$

where $\delta_{p,q,\pm} = \delta + p\omega_g + q\omega_e \pm kv$ are the position- and velocity-shifted detunings the force $\mathbf{F} = F\mathbf{e}_z$, $F = F(z,v)$ is represented as

$$F(z,v) = \hbar k\gamma \left[\left(2s_{1,-2,-} - \frac{1}{3}s_{1,0,+} \right) N_+ + (s_{0,-1,-} - s_{0,1,+})N_0 - \left(2s_{-1,2,+} - \frac{1}{3}s_{-1,0,-} \right) N_- \right.$$
$$\left. + \frac{1}{6}(s_{-1,0,-} - s_{1,0,+})(v + v^*) - \frac{\mathrm{i}}{6\gamma}(s_{-1,0,-}\delta_{-1,0,-} + s_{1,0,+}\delta_{1,0,+})(v - v^*) \right] , \tag{180}$$

where $N_0 = R^0_{g_0 g_0}$, $N_\pm = R^0_{g_{\pm 1} g_{\pm 1}}$ are the ground-state populations and $v = S^0_{g_{-1} g_1}$ is the ground-state coherence. Formula (180) defines the dipole radiation force in a low-intensity limit when the upper-state atomic populations can be neglected compared with the ground-state populations. Note that the ground-state coherence v is of the same order of magnitude as the ground-state populations.

The structure of the force $F(z,v)$ can be understood by separately considering the velocity dependence of the force at different atomic coordinates and the coordinate dependence at zero velocity. For small Larmor frequencies, $\omega_{\mathrm{L}g}, \omega_{\mathrm{L}e} \ll \gamma$, i.e. at small displacements from the origin of the trap, and at low velocities, $kv \ll \gamma$, the force can be found analytically. At a negative detuning, $\delta < 0$, the force reduces to the friction force and potential force,

$$F(z,v) = F(0,v) + F(z,0) \ ,$$

where the friction force $F(0,v)$ coincides with the friction force F defined by Eq. (101) and the potential force is (Jun et al., 1999a,b)

$$F(z,0) = -\frac{5}{11}\hbar k\gamma \frac{G|\delta|/\gamma^2}{(1 + \delta^2/\gamma^2)^2} \frac{(44/17)(3\omega_e - \omega_g)\tilde{\mu}^2 + (8\omega_e - 3\omega_g)\omega_g^2}{\omega_g^2 + \tilde{\mu}^2}$$
$$- \frac{5}{44}\hbar k\gamma \frac{G^2}{(1 + \delta^2/\gamma^2)^2} \frac{|\delta|\omega_g}{\omega_g^2 + \tilde{\mu}^2} \ . \tag{181}$$

Here the dimensionless saturation parameter G and the halfwidth of the two-photon resonance $\tilde{\mu}$ are defined by Eqs. (100) and (102). The position-dependent Zeeman shifts ω_g and ω_e are defined by Eqs. (178).

Fig. 27 shows the coordinate dependence of the potential force $F(z,0)$ near the origin of the trap. This dependence includes two different parts, the physical origin of which is the same as the origin of the two parts in the velocity-dependent force $F(0,v)$ defined by Eq. (101). The broad part in the coordinate dependence of the force $F(z,0)$ is due to the one-photon absorption (emission) processes

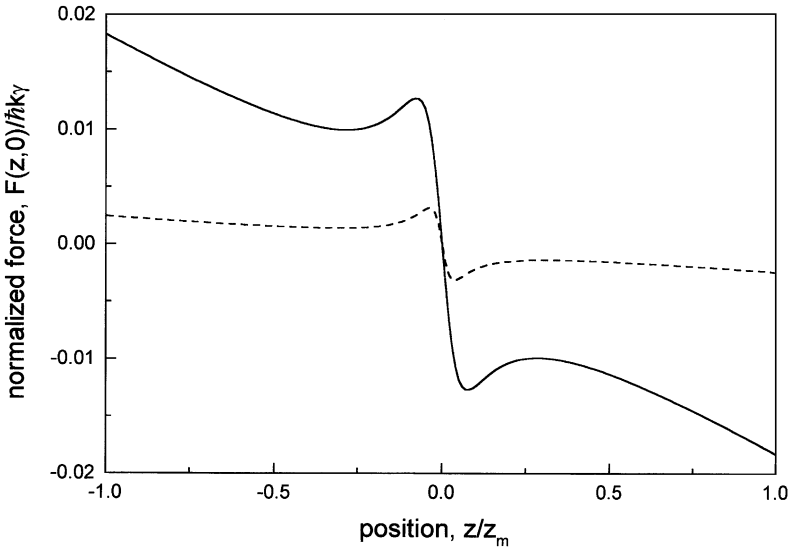

Fig. 27. Position-dependent force $F(z,0)$ on a (3+5)-level atom for the case $g_g = g_e$ at saturation parameter $G = 4$ and for detuning $\delta = -10\gamma$ (solid line) and $\delta = -20\gamma$ (dashed line). The displacement of the atom from the center of the trap is normalized on a characteristic distance $z_m = \hbar\gamma/\mu_B g a$ at which the Zeeman frequency shifts equal half of the natural linewidth.

being slightly perturbed by the two-photon processes. Narrow structure in the force $F(z,0)$ comes from the two-photon processes related to the ground-state coherence $v = S^0_{g_{-1}g_1}$. The ground-state coherence appears mainly due to the two-photon optical processes connecting two ground-state sublevels g_{-1}, g_1. At the origin of the trap, $z = 0$, the two-photon processes are effective at the velocities satisfying the two-photon resonance condition, $(\omega \pm kv) - (\omega \mp kv) \simeq 0$, i.e. at the velocities $v \simeq 0$. For motionless atom, $v = 0$, the two-photon resonance condition is $(\omega \pm \omega_g) - (\omega \mp \omega_g) \simeq 0$, and accordingly the two-photon processes are effective at the origin of the trap where $\omega_g \simeq 0$ and $z \simeq 0$. At low effective saturation the frequency width of the two-photon resonance is determined by the quantity $\delta\omega = \tilde{\mu}$. Accordingly, the velocity width of the two-photon resonance is about the value $\delta v = \tilde{\mu}/k$ and the coordinate width of the two-photon resonance is determined by the quantity $\delta z = \tilde{\mu}\hbar/\mu_B g_g a$.

7.3. Double-structure potential well

The potential force (181) can be integrated to give the potential energy for cold atoms,

$$U(z) = -\int_0^z F(z,0)\,\mathrm{d}z\ . \tag{182}$$

When the signs of the Zeeman shifts are chosen in such a way that the force $F(z,0)$ is an attractive force, the potential energy (182) describes the potential well of the MOT. The force $F(z) = F(z,0)$ owing to its two-component structure generally creates the two-component potential (Fig. 28). At a negative detuning, the MOT potential well may accordingly consist of a double-structure potential

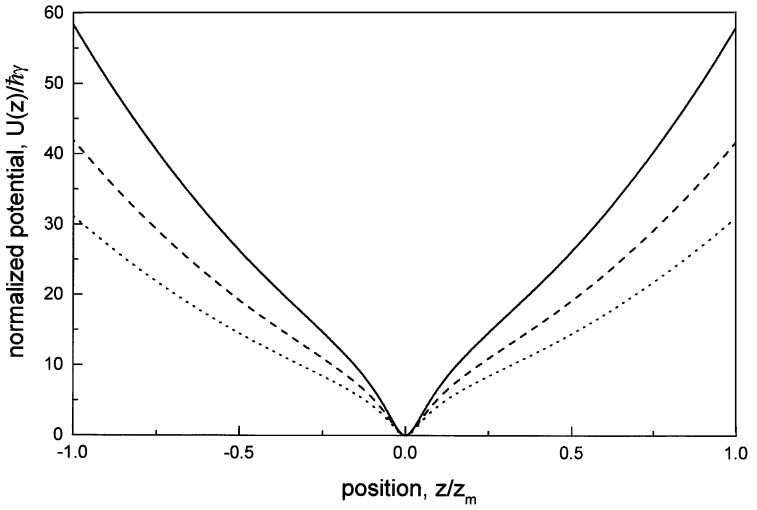

Fig. 28. Potential $U(z)$ of the magneto-optical trap for a (3+5)-level atom for the case $g_g = g_e$, at saturation parameter $G = 4$, and for detuning $\delta = -16\gamma$ (solid line), $\delta = -18\gamma$ (dashed line), and $\delta = -20\gamma$ (dotted line). Atomic coordinate is normalized on the characteristic distance $z_m = \hbar\gamma/\mu_B g a$.

well. The upper part of the well caused by the one-photon processes is relatively broad while the bottom part caused by the two-photon processes is relatively sharp.

The double structure of the MOT potential well can be illustrated by a simple model example of equal Zeeman frequency shifts, $\omega_g = \omega_e = \bar{\omega} = \mu_B g a z/\hbar > 0$, $g_g = g_e = g$, and for a case of large detuning, $|\delta| \gg \gamma$, when parameter $\tilde{\mu} \simeq \sqrt{17/33}G\gamma^2/4|\delta|$ and the potential force (181) has a simple form

$$F(z,0) \simeq -\frac{25}{11}\frac{G\gamma^3}{|\delta|^3}\hbar k\bar{\omega} - \frac{60}{17}\frac{\gamma}{|\delta|}\frac{\tilde{\mu}^2}{\bar{\omega}^2+\tilde{\mu}^2}\hbar k\bar{\omega}\,. \tag{183}$$

The force (183) creates the double-structure potential well described by the potential

$$U(z) \simeq \frac{25}{22}\frac{G\gamma^3}{|\delta|^3}\hbar\gamma\frac{k}{z_m}z^2 + \frac{5}{88}\hbar\gamma\frac{G^2\gamma^3}{|\delta|^3}kz_m \ln\left(1+\frac{528}{17}\frac{\delta^2}{G^2\gamma^2}\left(\frac{z}{z_m}\right)^2\right), \tag{184}$$

where $z_m = \hbar\gamma/\mu_B g a$ is a characteristic distance from the center of the trap at which the Zeeman frequency shifts equal to half of the natural linewidth. The first part of the potential (184) is basically due to the one-photon processes and the second one is due to the two-photon processes.

It is to be noted that the value of the sub-Doppler kinetic energy $k_B T$ according to Eq. (173) is generally of the same order of magnitude as the threshold energy that separates the sharp bottom part of the well from the upper broad part. For that reason, the spatial distribution of cold atoms in the double-structure potential well is generally a two-component distribution.

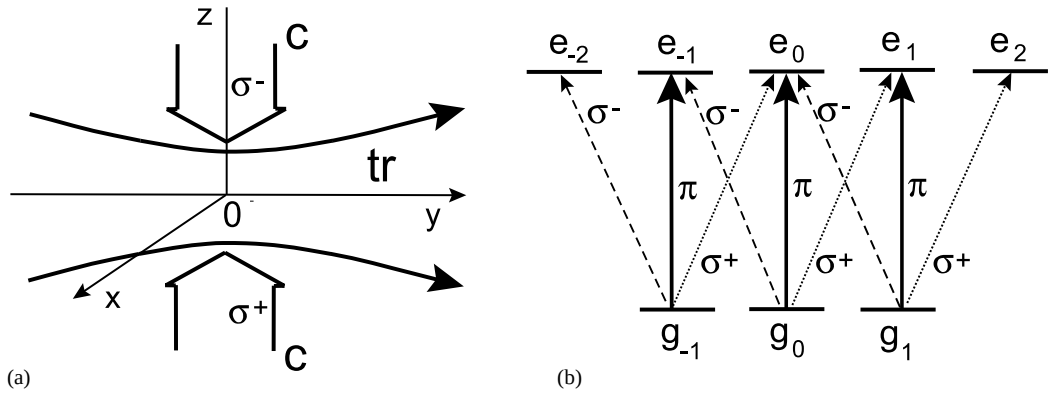

Fig. 29. Scheme of a far-off-resonance optical dipole trap (FORT) (a) and scheme of dipole transitions in a (3+5)-level atom in the FORT (b).

8. Optical dipole traps

In this section we consider an application of the kinetic approach to the dynamics of multilevel atoms in spatially inhomogeneous laser fields on an example of the optical dipole traps. Among different configurations of optical atom traps, of fundamental importance is a far-off-resonance optical dipole trap (FORT) based on a single focused far-detuned laser beam (Ashkin, 1978; Gordon and Ashkin, 1980; Chu et al., 1986; Miller et al., 1994; Corwin et al., 1999). The FORT produces a nearly conservative potential well for atoms, but incorporates an inevitable heating due to the photon recoil associated with the scattered laser light. Although the heating rate may be very small at very large detuning from the resonance, the photon recoil heating introduces an upper limit on the lifetime of atoms in the trap.

Below we present a kinetic theory analysis of the FORT composed of a single red-detuned trapping laser beam and describe a modification of the FORT which includes an additional red-detuned cooling laser field that prevents the heating due to the photon recoil (Garraway and Minogin, 2000). The theory of the FORT is considered below for a simplest realistic model of a (3+5)-level atom.

8.1. Single-beam optical dipole trap

We assume for definiteness that the FORT is based on a linearly polarized trapping laser beam (Fig. 29),

$$\mathbf{E}_{\mathrm{t}} = \mathbf{e}_0 E_{0\mathrm{t}}(\mathbf{r}) \cos(ky - \omega_{\mathrm{t}} t) \,, \tag{185}$$

which is defined by a unit polarization vector $\mathbf{e}_0 = \mathbf{e}_z$, spatially nonuniform amplitude $E_{0\mathrm{t}}(\mathbf{r})$, and the wave vector $\mathbf{k}_{\mathrm{t}} = k\mathbf{e}_y$ ($k = \omega_{\mathrm{t}}/c$). For a chosen interaction scheme, the atomic Hamiltonian has a standard form (117),

$$H = H_{\mathrm{a}} - \frac{\hbar^2}{2M}\nabla^2 - \mathbf{d} \cdot \mathbf{E}_{\mathrm{t}} \,, \tag{186}$$

where the Hamiltonian H_a describes the internal atomic states with energy levels $E_{g_0}, E_{g_{\pm1}}$ and $E_{e_0}, E_{e_{\pm1}}, E_{e_{\pm2}}$ and the last term describes the dipole interaction between the atom and the electric field $\mathbf{E}_t$ of the trapping laser beam.

Consider first the trapping potential produced by the laser beam. The semiclassical density matrix equations describing the dipole interaction of a (3+5)-level atom with the field (185) can be written in RWA according to Eqs. (61). Some types of equations for a considered scheme are as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_{-1}} &= \frac{\mathrm{i}\sqrt{3}}{2}\Omega(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}(ky-\Delta t)}\rho_{e_{-1}g_{-1}} + \text{c.c.} + \gamma\left(2\rho_{e_{-2}e_{-2}} + \rho_{e_{-1}e_{-1}} + \frac{1}{3}\rho_{e_0e_0}\right),\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} &= \mathrm{i}\Omega(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}(ky-\Delta t)}\rho_{e_0g_0} + \text{c.c.} + \gamma\left(\rho_{e_{-1}e_{-1}} + \frac{4}{3}\rho_{e_0e_0} + \rho_{e_1e_1}\right),\\
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0e_0} &= -\mathrm{i}\Omega(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}(ky-\Delta t)}(\rho_{g_0g_0} - \rho_{e_0e_0}) - \gamma\rho_{g_0e_0}\ .
\end{aligned} \tag{187}$$

The position-dependent Rabi frequency due to the trapping laser beam is defined here with respect to the most strong π-type dipole transition $|\alpha_g, F_g = 1, M_g = 0\rangle - |\alpha_e, F_e = 2, M_e = 0\rangle$,

$$\Omega(\mathbf{r}) = \frac{\langle\alpha_e 2\|d\|\alpha_g 1\rangle E_{0t}(\mathbf{r})}{\sqrt{30}\hbar}\ , \tag{188}$$

and the detuning is $\Delta = \omega_t - \omega_0$.

Eliminating an explicit time and position dependence in the above equations with the substitutions,

$$\rho_{g_{-1}e_{-1}} = \sigma_{g_{-1}e_{-1}}\mathrm{e}^{-\mathrm{i}(ky-\Delta t)},\quad \rho_{g_0e_0} = \sigma_{g_0e_0}\mathrm{e}^{-\mathrm{i}(ky-\Delta t)},\quad \rho_{g_1e_1} = \sigma_{g_1e_1}\mathrm{e}^{-\mathrm{i}(ky-\Delta t)},\ldots$$

and solving next the equations for a steady state, one can find the dipole radiation force on the atom according to Eq. (65). The force on a (3+5)-level atom in laser beam (185) includes, as usual, two partial forces, the gradient force and the radiation pressure force,

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_{gr} + \mathbf{F}_{rp}\ ,\\
\mathbf{F}_{gr} &= 2\hbar(\nabla\Omega(\mathbf{r}))\,\mathrm{Re}\left(\sigma_{g_0e_0} + \frac{\sqrt{3}}{2}(\sigma_{g_{-1}e_{-1}} + \sigma_{g_1e_1})\right),\\
\mathbf{F}_{rp} &= 2\hbar k\Omega\mathbf{e}_y\,\mathrm{Im}\left(\sigma_{e_0g_0} + \frac{\sqrt{3}}{2}(\sigma_{e_{-1}g_{-1}} + \sigma_{e_1g_1})\right)\ .
\end{aligned} \tag{189}$$

In the case of large negative detunings, $|\Delta| \gg \Omega(\mathbf{r})$, and for slowly moving atoms ($\mathbf{v} \approx 0$) the forces are

$$\begin{aligned}
\mathbf{F}_{gr} &= \frac{30}{17}\hbar\frac{\nabla\Omega^2(\mathbf{r})}{|\Delta|}\ ,\\
\mathbf{F}_{rp} &= \frac{30}{17}\hbar k\,\gamma\mathbf{e}_y\frac{\Omega^2(\mathbf{r})}{\Delta^2}\ .
\end{aligned} \tag{190}$$

The gradient force produces the potential well for the atoms that differs from that for a two-level atom (79) by a numerical factor only,

$$U(\mathbf{r}) = -\int_{-\infty}^{\mathbf{r}}\mathbf{F}_{gr}\cdot\mathrm{d}\mathbf{r} = -\frac{30}{17}\hbar\frac{\Omega^2(\mathbf{r})}{|\Delta|}\ . \tag{191}$$

The radiation pressure force produces an additional asymmetric potential well which can be neglected at large detuning.

The lifetime τ of the atom in the FORT is always restricted by the diffusion heating according to a general estimation,

$$\tau \simeq \frac{2MU_0}{D(0)} ,$$

where U_0 is the depth of the potential well (191) and $D(0)$ is the momentum diffusion coefficient at zero velocity. Taking into account the fact that according to Eq. (191) the depth of the potential well is $U_0 \simeq \hbar\Omega^2(0)/|\Delta|$ and according to Eq. (152) the diffusion coefficient is $D(0) \simeq \hbar k^2\gamma\Omega^2/\Delta^2$, one can see that the lifetime of the atom in the FORT is about $\tau \simeq \omega_r^{-1}(|\Delta|/\gamma)$, where ω_r is the recoil frequency. When the detuning increases the lifetime of the atom in the FORT thus grows but with a simultaneous decrease in the depth of the potential well.

8.2. Single-beam dipole trap with superimposed laser cooling

It was proposed that the heating mechanism in the FORT might be suppressed by adding the cooling laser field to a focused trapping laser beam (Gordon and Ashkin, 1980; Chu et al., 1986). The experiments with different types of the cooling laser fields have proved that the addition of the cooling field can increase the lifetime of the atoms and even the atomic density in the trap (Lee et al., 1996; Boiron et al., 1998).

The addition of the cooling field may have a profound effect on all the basic parameters of the FORT since the cooling field may strongly influence the atomic populations and coherences. Typically any cooling laser field operates at a detuning less than the detuning of the trapping laser beam. The cooling field may thus be responsible not only for perturbation of the steady-state internal atomic state but the perturbation of the trapping potential as well. The FORT with an additional cooling laser field was theoretically discussed for the simplest model of a two-level dipole interaction scheme (Gordon and Ashkin, 1980). A two-level model has, however, a very limited connection with real experimental techniques which typically explore multilevel dipole interaction schemes. Physically, there is a major difference between the models of a two-level atom and a multilevel atom in applications related to the trapping and cooling atoms. In a two-level model both trapping and cooling fields excite atoms on the same atomic transition. As a result, for a two-level atomic scheme in the FORT, the depth of the potential well and the cooling limit have generally the same order of magnitude defined by the Doppler temperature. In multilevel atomic schemes the trapping and cooling laser fields can explore principally different atomic transitions. The trapping field can produce a potential well due to the one-photon transitions while the cooling laser field can cool atoms down to the sub-Doppler temperatures due to the two-photon transitions. It is thus important that in multilevel atomic schemes the optical processes used for trapping atoms in the FORT and those for the sub-Doppler cooling can have different physical origins. The use of the different optical processes for trapping and cooling multilevel atoms thus raises questions on basic parameters of the trap and the lifetime of atoms achievable in the FORT with superimposed sub-Doppler cooling process.

We discuss below the scheme of the FORT for a (3+5)-level atom interacting with the trapping laser field (185) and the cooling laser field (82).

8.2.1. Quantum-kinetic equations

To analyze the operation of the trap at large detunings, it is sufficient to consider the quantum-kinetic equations at small optical saturation. We will accordingly take into account one- and two-photon optical processes, and neglect higher-order optical processes. To simplify the consideration of the equations of motion, we additionally neglect the spatial variation in the trapping laser beam amplitude $E_{0t}(\mathbf{r})$. The effect of the spatially inhomogeneous trapping laser beam will be taken into account separately.

A set of the atomic density matrix equations in the Wigner representation and RWA describing an interaction of a (3+5)-level atom with laser waves (185) and (82) of constant amplitude at weak optical saturation can be written according to general equations (134). Some types of the quantum-kinetic equations for a considered scheme are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_{-1}} = \frac{\mathrm{i}\sqrt{3}}{2}\Omega \mathrm{e}^{-\mathrm{i}(ky-\Delta t)}\rho^{(+)}_{e_{-1}g_{-1}} + \mathrm{i}\theta \mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(-)}_{e_{-2}g_{-1}} + \frac{\mathrm{i}\theta}{\sqrt{6}}\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_0g_{-1}} + \mathrm{c.c}\ .$$
$$+\gamma\left(2\langle\rho^{-1}_{e_{-2}e_{-2}}\rangle + \langle\rho^{0}_{e_{-1}e_{-1}}\rangle + \frac{1}{3}\langle\rho^{1}_{e_0e_0}\rangle\right)\ ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0g_0} = \mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}(ky-\Delta t)}\rho^{(+)}_{e_0g_0} + \frac{\mathrm{i}\theta}{\sqrt{2}}\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(-)}_{e_{-1}g_0} + \frac{\mathrm{i}\theta}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_1g_0} + \mathrm{c.c}\ .$$
$$+\gamma\left(\langle\rho^{-1}_{e_{-1}e_{-1}}\rangle + \frac{4}{3}\langle\rho^{0}_{e_0e_0}\rangle + \langle\rho^{1}_{e_1e_1}\rangle\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_{-1}g_1} = \frac{\mathrm{i}\theta}{\sqrt{6}}(\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_0g_1} - \mathrm{e}^{-\mathrm{i}kz-\mathrm{i}\delta t}\rho^{(-)}_{g_{-1}e_0})$$
$$+\gamma\left(\sqrt{\frac{2}{3}}\langle\rho^{-1}_{e_{-2}e_0}\rangle + \langle\rho^{0}_{e_{-1}e_1}\rangle + \sqrt{\frac{2}{3}}\langle\rho^{1}_{e_0e_2}\rangle\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{g_0e_0} = -\mathrm{i}\Omega \mathrm{e}^{-\mathrm{i}(ky-\Delta t)}(\rho^{(-)}_{g_0g_0} - \rho^{(+)}_{e_0e_0}) + \frac{\mathrm{i}\theta}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(-)}_{e_{-1}e_0} + \mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{e_1e_0})$$
$$-\frac{\mathrm{i}\theta}{\sqrt{6}}(\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(-)}_{g_0g_{-1}} + \mathrm{e}^{\mathrm{i}kz+\mathrm{i}\delta t}\rho^{(+)}_{g_0g_1}) - \gamma\rho_{g_0e_0}\ . \tag{192}$$

In the above equations θ and δ are the Rabi frequency for the cooling laser field and the detuning of the cooling field,

$$\theta = \frac{\langle\alpha_e 2\|d\|\alpha_g 1\rangle E_{0\mathrm{c}}}{2\sqrt{5}\hbar}, \quad \delta = \omega_{\mathrm{c}} - \omega_0\ . \tag{193}$$

Note that the Rabi frequency is defined with respect to the strongest σ-type dipole transition as in Eq. (90).

8.2.2. Quasiclassical description

The reduction of the initial quantum-kinetic equations to the quasiclassical Fokker–Planck equation can be done by a standard procedure of Section 5.1. Recalling that the initial equations include the

terms describing only one- and two-photon processes one can first introduce the substitutions which take into account only one- and two-photon processes:

$$\rho_{g_{-1}e_{-2}} = \sigma_{g_{-1}e_{-2}} e^{ikz+i\delta t}, \quad \rho_{g_{-1}e_{-1}} = \sigma_{g_{-1}e_{-1}} e^{-iky+i\Delta t}, \quad \rho_{g_{-1}e_0} = \sigma_{g_-e_0} e^{-ikz+i\delta t} ,$$

$$\rho_{g_{-1}g_0} = \sigma^1_{g_{-1}g_0} e^{-ik(y+z)+i(\Delta-\delta)t} + \sigma^2_{g_{-1}g_0} e^{ik(y-z)-i(\Delta-\delta)t}, \quad \rho_{g_{-1}g_1} = \sigma_{g_{-1}g_1} e^{-2ikz}, \dots .$$

After that the initial microscopic equations reduce to the equations which do not include an explicit time and coordinate dependence.

Next, we apply a standard procedure of the transition to the quasiclassical description and find the force and momentum diffusion tensor entering the Fokker–Planck equation (148) as

$$F_{rp} = 2\hbar k\theta \,\mathrm{Im}\left[(S^0_{e_2g_1} - S^0_{e_{-2}g_{-1}}) + \frac{1}{\sqrt{2}}(S^0_{e_1g_0} - S^0_{e_{-1}g_0}) + \frac{1}{\sqrt{6}}(S^0_{e_0g_{-1}} - S^0_{e_0g_1})\right], \tag{194}$$

$$\begin{aligned} D_{ii} = {} & \hbar^2k^2\gamma\left[\alpha^\sigma_{ii}\left(R^0_{e_{-2}e_{-2}} + \frac{1}{2}R^0_{e_{-1}e_{-1}} + \frac{1}{3}R^0_{e_0e_0} + \frac{1}{2}R^0_{e_1e_1} + R^0_{e_2e_2}\right)\right. \\ & \left. + \alpha^\pi_{ii}\left(\frac{1}{2}R^0_{e_{-1}e_{-1}} + \frac{2}{3}R^0_{e_0e_0} + \frac{1}{2}R^0_{e_1e_1}\right)\right] \\ & + \delta_{iz}\hbar^2k^2\theta \,\mathrm{Im}\left[(T^1_{g_1e_2} + T^1_{e_{-2}g_{-1}}) + \frac{1}{\sqrt{2}}(T^1_{g_0e_1} + T^1_{e_{-1}g_0}) + \frac{1}{\sqrt{6}}(T^1_{g_{-1}e_0} + T^1_{e_0g_1})\right], \end{aligned} \tag{195}$$

where the coefficients α^σ_{ii}, α^π_{ii} are defined by Eq. (153). Note, that in the above derivation the laser beams are considered as plane light waves. The dipole radiation force in a field of plane waves (194) is accordingly called the radiation pressure force, $\mathbf{F} = \mathbf{F}_{rp} = \mathbf{e}_z F_{rp}$.

8.2.3. Dipole gradient force

The derivation of the gradient force $\mathbf{F}_{gr}$ associated with the gradient of the trapping laser beam amplitude $\mathbf{E}_{0t} = \mathbf{E}_{0t}(\mathbf{r})$ can be done in a way that generalizes the procedure considered in the preceding subsection. Representing the trapping laser beam amplitude in the form of a Fourier expansion,

$$\mathbf{E}_{0t}(\mathbf{r}) = (2\pi)^{-3/2}\int \mathbf{E}_{0t}(\mathbf{q})\, e^{iqr}\, d^3q , \tag{196}$$

one should introduce into the Eqs. (192) the following substitutions:

$$i\Omega(\mathbf{r})\rho_{ab}(\mathbf{p}) \to (2\pi)^{-3/2}\int i\Omega(\mathbf{q}) e^{iqr}\rho_{ab}(\mathbf{p} + \tfrac{1}{2}\hbar\mathbf{q})\, d^3q . \tag{197}$$

For a laser beam amplitude that varies in space on a scale which is large compared to the size of the atomic wave packet, it is sufficient to expand the density matrix elements to a first order in the small momentum $\hbar\mathbf{q}$,

$$\rho_{ab}(\mathbf{p} + \tfrac{1}{2}\hbar\mathbf{q}) \simeq \rho_{ab}(\mathbf{p}) + \tfrac{1}{2}\hbar\mathbf{q}\frac{\partial}{\partial\mathbf{p}}\rho_{ab}(\mathbf{p}) . \tag{198}$$

This transforms the terms in the equations as

$$i\Omega(\mathbf{r})\rho_{ab}(\mathbf{p}) \to i\Omega(\mathbf{r})\rho_{ab}(\mathbf{p}) + \frac{\hbar}{2}\nabla\Omega(\mathbf{r})\frac{\partial}{\partial\mathbf{p}}\rho_{ab}(\mathbf{p}) \ . \tag{199}$$

When the substitutions (199) are taken into account the Fokker–Planck equation includes, in the above approximation, the total dipole radiation force as a sum of the gradient force and the radiation pressure force (194), $\mathbf{F} = \mathbf{F}_{\rm gr} + \mathbf{F}_{\rm rp}$.

The gradient force is then determined by the steady-state optical coherences as

$$\mathbf{F}_{\rm gr} = 2\hbar(\nabla\Omega(\mathbf{r}))\,{\rm Re}(S^0_{g_0e_0} + \tfrac{\sqrt{3}}{2}(S^0_{g_{-1}e_{-1}} + S^0_{g_1e_1})) \ . \tag{200}$$

Note that a formal mathematical expression for this part of the total radiation force coincides with that defined by Eq. (189). The explicit expressions given by Eqs. (200) and (189) are generally different since the functions S^0_{ab} describe the interaction of the atom with the total laser field, while the functions σ_{ab} describe the interaction of the atom with the trapping laser beam only.

8.2.4. Optical potential depth and kinetic energy

We next consider a regime characterized by two important limits. That is, we consider large detunings,

$$|\Omega|, |\delta| \gg \gamma \ , \tag{201}$$

and low optical saturation,

$$s_{\rm t} = \frac{\Omega^2}{\Delta^2} \ll 1, \quad s_{\rm c} = \frac{\theta^2}{\delta^2} \ll 1 \ , \tag{202}$$

when the one-photon and two-photon processes play a dominant role in the time evolution of the atomic density matrix elements. In addition to the above approximations, we restrict our treatment to the case of slowly moving atoms,

$$\eta_y = \frac{kv_y}{\gamma} \ll 1, \quad \eta_z = \frac{kv_z}{\gamma} \ll 1 \ . \tag{203}$$

Under the above conditions the equations for the functions R^0_{aa}, S^0_{ab} and T^1_{ab}, which follow from Eqs. (192) expanded to the first order in the photon momentum, have a simple analytical solution. When solving the last equations one can note that the efficiencies of the two-photon optical processes between the ground-state magnetic sublevels depend crucially on two parameters μ and ν,

$$\mu = \frac{5}{6}\frac{\theta^2}{\delta^2}\gamma, \quad \nu = \frac{\Omega^2}{\Delta^2}\gamma \ , \tag{204}$$

which define the frequency widths of the two-photon resonance structures related to the ground-state coherences $\rho_{g_{-1}g_1}$ and $\rho_{g_{-1}g_0}, \rho_{g_0g_1}$. Physically, the origin of these two-photon frequency widths is the same as for a (3+5)-level atom in the field of a $\sigma^+ - \sigma^-$ laser field configuration (Section 3.3).

Since the two-photon processes induced by the cooling laser field are of importance only for the process of sub-Doppler cooling of atoms in the trap, the physical meaning of parameters μ and ν allows us to introduce two conditions necessary for separating the cooling and trapping processes: i.e.

$$\nu \ll \mu \ll \gamma \ . \tag{205}$$

The left inequality guarantees that the trapping field does not influence the two-photon cooling process, while the right inequality is needed for the two-photon friction coefficient to be greater than the one-photon friction coefficient.

Under conditions (205), and in lowest order in the small parameters s_t, s_c, η_y and η_z, the low-saturation, low-velocity solutions give the one-photon coherences entering Eq. (200) as

$$S^0_{g_{-1}e_{-1}} = \frac{-i\sqrt{3}\Omega/2}{\gamma + i\Delta} N_-, \quad S^0_{g_0e_0} = \frac{-i\Omega}{\gamma + i\Delta} N_0, \quad S^0_{g_1e_1} = \frac{-i\sqrt{3}\Omega/2}{\gamma + i\Delta} N_+ \ . \tag{206}$$

In the above the steady-state ground-state populations $N_- = R^0_{g_{-1}g_{-1}}$, $N_0 = R^0_{g_0g_0}$ and $N_+ = R^0_{g_1g_1}$, connected to the ground-state two-photon coherence $\rho_{g_{-1}g_1}$, are

$$\begin{aligned} N_- &= \frac{1}{2\tilde{\Delta}} \left(\frac{13}{12} \frac{\theta^4}{\delta^2} - \frac{15}{6} \frac{\theta^2}{|\delta|} k v_z + 9k^2 v_z^2 \right) \ , \\ N_0 &= \frac{2}{\tilde{\Delta}} \left(\frac{1}{6} \frac{\theta^4}{\delta^2} + k^2 v_z^2 \right) , \\ N_+ &= \frac{1}{2\tilde{\Delta}} \left(\frac{13}{12} \frac{\theta^4}{\delta^2} + \frac{15}{6} \frac{\theta^2}{|\delta|} k v_z + 9k^2 v_z^2 \right) , \end{aligned} \tag{207}$$

and the common denominator is

$$\tilde{\Delta} = \frac{17}{12} \frac{\theta^4}{\delta^2} + 11k^2 v_z^2 \ . \tag{208}$$

The other steady-state coherences and populations entering Eqs. (194) and (195) are defined by similar equations,

$$S^0_{g_{-1}e_{-2}} = \frac{-i\theta}{\gamma + i\delta} N_-, \quad S^0_{g_{-1}e_0} = \frac{-i\theta/\sqrt{6}}{\gamma + i\delta} N_-, \quad S^0_{g_0e_{-1}} = \frac{-i\theta/\sqrt{2}}{\gamma + i\Delta} N_0, \ldots \ . \tag{209}$$

The atomic coherences, together with the values of the ground-state populations, give a new value of the gradient force at zero velocity and negative detuning,

$$\mathbf{F}_{gr} = \frac{55}{68} \hbar \frac{\nabla \Omega^2(\mathbf{r})}{|\Delta|} \ . \tag{210}$$

In a corresponding way they give a new value of the potential,

$$U(\mathbf{r}) = -\int_{-\infty}^{\mathbf{r}} \mathbf{F}_{gr} \cdot \mathbf{dr} = -\frac{55}{68} \hbar \Omega(\mathbf{r}) \frac{\Omega(\mathbf{r})}{|\Delta|} \ , \tag{211}$$

which is three times less than the unperturbed potential (191). The reduced value of the gradient force and the potential in the total laser field is naturally explained by the dominant role of the cooling field in producing atomic populations in accordance with the left condition in Eq. (205). While the trapping field alone produces the ground-state atomic populations at zero velocity as $N_- = N_+ = 13/34$, $N_0 = 4/17$, the cooling field redistributes the populations to the values $N_- = N_+ = 9/22$, $N_0 = 4/22$. This redistribution, "multiplied" by the relative strengths of the dipole transitions, gives the decrease in the potential according to Eq. (211).

Substitution of the steady-state atomic coherences and ground-state populations into Eq. (194) gives an explicit expression for the radiation pressure force in a low-saturation, low-velocity approximation. Since the above analytical expressions (207) and (209) neglect a weak velocity dependence due to the one-photon (Doppler) processes, the radiation pressure force is described in our analysis only near zero velocity, i.e. in the region $|v| \lesssim \mu/k$. In the case of a red-detuned cooling field, where $\delta < 0$, the low-velocity part of the radiation pressure force reduces to the cooling force. If force (194) taken at red detuning and at low velocities is represented in a standard form of the friction force (162) the friction coefficient is $\beta = (120/17)\gamma\omega_{\rm r}/|\delta|$, where $\omega_{\rm r} = \hbar k^2/2M$ is a recoil frequency. The diffusion coefficient at zero velocity, $D(0)$, can be estimated as before to give $D(0) = (46/17)\hbar^2 k^2 \gamma\theta^2/\delta^2$.

The diffusion and friction coefficients jointly define the average kinetic energy and temperature of the atoms in the trap which according to Eq. (165) is

$$E = k_{\rm B}T = \frac{23}{30}\frac{\hbar\theta^2}{|\delta|} \ . \tag{212}$$

8.2.5. Conditions for stable trapping

Assuming now that kinetic energy (212) of cold trapped atoms is much less than the depth $U(0)$ of the potential well (211) one can get a sufficient condition for stable atomic trapping (Garraway and Minogin, 2000),

$$\frac{\theta^2}{|\delta|} \ll \frac{\Omega^2(0)}{|\Delta|} \ . \tag{213}$$

Comparing the last condition with the condition for deep laser cooling (205) one can see that both conditions are satisfied if the detuning of the trapping field is much larger than that of the cooling field,

$$|\Delta| \gg |\delta| \ . \tag{214}$$

The above two conditions, defined by Eqs. (205) and (213), justify the idea of a stable dipole trap. The stable atomic trapping in the optical dipole trap can thus be achieved when the trapping field has no effect on the two-photon cooling process, and the cooling field does not change the structure of the trapping potential but changes only the numerical value of the trapping potential well.

The lifetime of the atoms in the trap associated with the diffusive heating can finally be estimated as

$$\tau = \varpi {\rm e}^{U(0)/E} \tag{215}$$

where ϖ is the oscillation frequency of an atom in the trap and the Boltzmann factor $U(0)/E$ considerably exceeds unity.

The above evaluations of the conditions necessary and sufficient for stable atomic trapping can be illustrated by a specific example. Assume that the Rabi frequency and the detuning for a cooling field are accordingly $\theta = \gamma$ and $|\delta| = 20\gamma$ and the Rabi frequency and the detuning for the trapping field are $\Omega = 10^2\gamma$ and $|\Delta| = 10^4\gamma$. For these parameters the one-photon widths $\nu = 10^{-4}\gamma$ and $\mu = 2.5 \times 10^{-3}\gamma$ satisfy conditions (205) for deep laser cooling. In their turn, the kinetic energy (212), estimated as $E = 0.04\hbar\gamma$, and the potential well depth, estimated according to Eq. (211) as $U(0) = 0.8\hbar\gamma$, satisfy

the sufficient condition (213) for stable atomic trapping giving for the Boltzmann factor a sufficiently large value $U(0)/E = 20$.

9. Conclusion

We conclude by emphasizing that the density matrix approach is a powerful technique which can be successfully applied to many problems of atomic excitation and dynamics in laser fields. We do hope that the approach described and the examples presented in this paper will help the reader to extend the application of the technique to new exciting problems.

We finally stress that in all the considered problems we mainly paid attention to basic theoretical procedures and the features of atomic dynamics specific to multilevel interaction schemes. Following the basic purpose of this paper to describe the applications of the density matrix technique to the problems of atomic dynamics we referred only to basic theoretical papers in the field. For a latest review of achievements in the field of control of atomic dynamics and motion by laser light we refer the reader to review articles and books (Dowling and Gea-Banacloche, 1996; Jessen and Deutsch, 1996; Berman, 1997; Adams and Riis, 1997; Grimm et al., 1999; Metcalf and van der Straten, 1999; Balykin et al., 2000).

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